Appendix 1: Principle of Total Probability

Definition:

- Set $H = \{A_1, \ldots, A_n\}$ where $A_j \subseteq S$, $j = 1, \ldots, n$, is a **partition** of the set S if

  1. $(\forall i, j)\left(i \neq j \Rightarrow A_i \cap A_j = \varnothing\right)$

  2. $\bigcup_{j=1}^{n} A_j = S$ •

In particular, if $H \subseteq \boldsymbol{U}(\Omega)$ and $S \in \boldsymbol{U}(\Omega)$, where $\boldsymbol{U}(\Omega)$ denotes the event algebra from a probability space $(\Omega, \boldsymbol{U}(\Omega), \Pr)$, then

$$
\begin{aligned}
\Pr(S) &= \Pr\left(\bigcup_{j=1}^{n} A_j\right) = \Pr\left(\bigcup_{j=1}^{n-1} A_j\right) + \Pr\left(A_n\right) - \Pr\left(\bigcup_{j=1}^{n-1} A_j \cap A_n\right) \\
&= \Pr\left(\bigcup_{j=1}^{n-1} A_j\right) + \Pr\left(A_n\right)
\end{aligned}
$$

and by induction

$$\Pr(S) = \sum_{j=1}^{n} \Pr(A_j)$$

The partition $H = \{A_1, \ldots, A_n\}$ is **positive** if $\Pr(A_j) > 0$, $j = 1, \ldots, n$.

Corollary (The principle of total probability):

- Let $H = \{A_1, \ldots, A_n\}$, $(\Pr(A_j) > 0, j = 1, \ldots, n)$ be a positive partition of the event space Ω, and $B \in \boldsymbol{U}(\Omega)$. Then

  $$\Pr(B) = \sum_{j=1}^{n} \Pr\left(B \middle| A_j\right) \Pr\left(A_j\right)$$ •

Proof:

- $H_B = \{(B \cap A_1), \ldots, (B \cap A_n)\}$

  is obviously a partition of the event *B*. Moreover, since

  $$B = \bigcup_{j=1}^{n} \left(B \cap A_j\right)$$

  it follows that

  $$\Pr\left(B\right) = \sum_{j=1}^{n} \Pr\left(B \cap A_j\right).$$

  Hence, by substituting the formula of conditional probability,

  $$\Pr(B \cap A_j) = \Pr\left(B \middle| A_j\right) \Pr\left(A_j\right),$$

  we have

  $$\Pr(B) = \sum_{j=1}^{n} \Pr\left(B \middle| A_j\right) \Pr\left(A_j\right).$$

•

In special case, when n = 2,

$$\Pr\left(B\right) = \Pr\left(B \middle| A\right) \Pr\left(A\right) + \Pr\left(B \middle| A^c\right) \Pr\left(A^c\right).$$