$$\frac{1}{v_\ell}\frac{\partial \phi_\ell}{\partial t} = \sum_{j=1}^{G} \phi_j \left\{ \nu_j (1-\beta_j) \Sigma_{fj} \chi_{p\ell} + \Sigma_{sj} f_s (E_j \to E_\ell) \right\} + \nabla \cdot D_\ell \nabla \phi_\ell + \sum_{i=1}^{N} \lambda_i C_i \chi_{i\ell} - (\Sigma_{a\ell} + \Sigma_{s\ell}) \phi_\ell$$

$$\frac{\partial C_K}{\partial t} = -\lambda_K C_K + \sum_{j=1}^{G} \beta_{Kj} \nu_j \Sigma_{fj} \phi_j$$

<u>OR</u>

$$\begin{bmatrix} \frac{\partial \phi_1}{\partial t} \\ \vdots \\ \frac{\partial \phi_G}{\partial t} \\ \frac{\partial C_1}{\partial t} \\ \vdots \\ \frac{\partial C_N}{\partial t} \end{bmatrix} = \begin{array}{c} \begin{array}{cc} j \longrightarrow & i \longrightarrow \end{array} \\ \left[\begin{array}{ccc|ccc} a_{11} & a_{12} \cdots & a_{1G} & a_{1,G+1} & \cdots & a_{1,G+N} \\ & a_{\ell j} & & & & \\ a_{G1} & \cdots & a_{GG} & a_{G,G+1} & \cdots & a_{G,G+N} \\ \hline a_{G+1,1} & \cdots & a_{G+1,G} & a_{G+1,G+1} & \cdots & a_{G+1,G+N} \\ & a_{G+K,j} & & & a_{G+K,G+i} & \\ a_{G+N,1} & \cdots & a_{G+N,G} & a_{G+N,G+1} & \cdots & a_{G+N,G+N} \end{array}\right] \end{array} \begin{array}{c} \ell \downarrow \\ \\ K \downarrow \end{array} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_G \\ C_1 \\ \vdots \\ C_N \end{bmatrix} \qquad \underline{\underline{\text{OR}}} \quad \frac{\partial \underline{\Psi}}{\partial t} = \underline{\underline{A}}\,\underline{\Psi}$$

$$\frac{\partial \underline{\Psi}}{\partial t} = \underline{\underline{A}} \qquad \underline{\Psi}$$

| | FLUX GROUP NUMBER 1 | 2 | j | G | PRECURSOR GROUP NUMBER 1 | 2 … i | N | |
|---|---|---|---|---|---|---|---|---|
| 1 | $V_1\{\nu_1(1-\beta_1)\Sigma_{f_1}\chi_{p1} + \Sigma_{s1} f_s^{11} - \Sigma_{a1} - \Sigma_{s1} + \nabla \cdot D_1 \nabla\}$ | $V_1\{\nu_2(1-\beta_2)\Sigma_{f_2}\chi_{p1} + \Sigma_{s2} f_s^{21}\}$ | … | $V_1\{\nu_G(1-\beta_G)\Sigma_{fG}\chi_{p1} + \Sigma_{sG} f_s^{G1}\}$ | $V_1\lambda_1 C_1 \chi_{11}$ | … | $V_1\lambda_N C_N \chi_{N1}$ | 1 |
| 2 | $V_2\{\nu_1(1-\beta_1)\Sigma_{f_1}\chi_{p2} + \Sigma_{s1} f_s^{12}\}$ | | | | $V_2\lambda_1 C_1 \chi_{12}$ | | | ⋮ |
| $\ell$ | ⋮ | | $V_\ell\{\nu_\ell(1-\beta_\ell)\Sigma_{f_\ell}\chi_{p\ell} + \Sigma_{s\ell} f_s^{\ell\ell} - \Sigma_{a\ell} - \Sigma_{s\ell} + \nabla \cdot D_\ell \nabla\}$ | | ⋮ | $V_\ell \lambda_i C_i \chi_{i\ell}$ | | $\ell$ |
| G | $V_G\{\nu_1(1-\beta_1)\Sigma_{f_1}\chi_{pG} + \Sigma_{s1} f_s^{1G}\}$ | — | — | $V_G\{\nu_G(1-\beta_G)\Sigma_{fG}\chi_{pG} + \Sigma_{sG} f_s^{GG}\}$ | $V_G\lambda_1 C_1 \chi_{1G}$ | … | $V_G\lambda_N C_N \chi_{NG}$ | G |
| 1 | $\beta_{11}\nu_1\Sigma_{f_1}$ | $\beta_{12}\nu_2\Sigma_{f_2}$ | … | $\beta_{1G}\nu_G\Sigma_{fG}$ | $-\lambda_1$ | | 0 | 1 |
| k | | | $\beta_{kj}\nu_j\Sigma_{fj}$ | | | $-\lambda_i$ | | k |
| N | $\beta_{N1}\nu_1\Sigma_{f_1}$ | $\beta_{N2}\nu_2\Sigma_{f_2}$ | … | $\beta_{NG}\nu_G\Sigma_{fG}$ | 0 | | $\lambda_N$ | G |

Solutions to $\frac{\partial \underline{\Psi}}{\partial t} = \underline{\underline{A}}\,\underline{\Psi}$ or $\dot{\underline{\Psi}} = \underline{\underline{A}}\,\underline{\Psi}$

(1) $\underline{\Psi} = \underline{\underline{\Omega}}\,\underline{\Phi}(\underline{r},t)$ where $\underline{\underline{\Omega}} = \begin{bmatrix} \sum_{l=1}^{G+N} \psi_{1l} e^{\alpha_l t} & & & 0 \\ & \ddots & & \\ & & \sum_{l=1}^{G+N} \psi_{il} e^{\alpha_l t} & \\ 0 & & & \ddots \;\; \sum_{l=1}^{G+N} \psi_{G+N,l} e^{\alpha_l t} \end{bmatrix}$

$\Downarrow$

$$\dot{\underline{\Phi}} = \underline{\underline{\Omega}}^{-1}(\underline{\underline{A}} - \dot{\underline{\underline{\Omega}}}\,\underline{\underline{\Omega}}^{-1})\,\underline{\underline{\Omega}}\,\underline{\Phi}$$

$\Downarrow$

$$? \; \dot{\underline{\Phi}} = \left[\sum_l (\underline{\underline{A}} - \alpha_l \underline{\underline{I}})\,\underline{\Psi}_l\right] \underline{\Phi}(\underline{r},t)$$

$$= \underbrace{\begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & 0 & 1 & \\ & & & \ddots \; 1 \end{bmatrix}}_{=\underline{\underline{I}}} \begin{bmatrix} \sum_{l=1}^{G+N} \psi_{1l} e^{\alpha_l t} \\ \vdots \\ \sum_1 \psi_{G+N,l} e^{\alpha_l t} \end{bmatrix} \Rightarrow \underline{\Psi} = \sum_l \underline{\Psi}_l e^{\alpha_l t}$$

(2) $\underline{\Psi} = e^{\underline{\underline{A}}t}\,\underline{\Psi}(0)$ where $e^{\underline{\underline{A}}t} = \underline{\underline{I}} + \underline{\underline{A}}t + \frac{\underline{\underline{A}}^2}{2!}t^2 + \frac{\underline{\underline{A}}^3}{3!}t^3 + \ldots$