Final 2005 EP4D03

1. For $I(x) = I_0 e^{-\Sigma x}$

a) $\frac{\partial I}{\partial x} = -\Sigma I$ where $\Sigma = \sigma N$

σ ← microscopic cross section; N ← # target nuclei/cm^3

$I = v\, n$

v ← speed; n ← # of neutrons/cm^3

b) Equation has a lot of approximations in it.

i) assumes neutron is removed when it interacts - ie is absorbed or scatters away. In fact multiple scatters can direct the neutron back into the beam. Correct via buildup factor or model diffusion.

ii) σ and N can be functions of x. Hence, in general $\Sigma = fn(x)$. Can solve numerically.

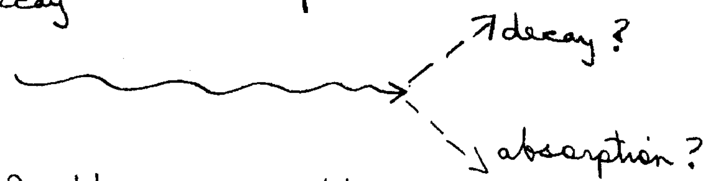
iii) σ (hence Σ) is a function of E in general. So which E to use?

iv) n is also a fn(E) and neutrons will lose energy when scattering. Need to model this.

c) To be more precise, use the multigroup space dependent diffusion equations. Or transport equations, - since we need to explicitly account for E, multiple scatters, $\Sigma(x)$ etc. Maybe even use 3-D.

2.

$$dn = \underbrace{-\lambda n\,dt}_{\text{decay}} - \underbrace{\Sigma_a n\,dx}_{\text{absorption}} \quad , \quad dx = v\,dt$$



$$\therefore\ dn = -\lambda n\,dt - \Sigma_a n v\,dt = -(\lambda + \Sigma_a v)\,dt$$

$$\therefore\ n = n(0)\,e^{-(\lambda + \Sigma_a v)t}$$

$$\text{ratio of decay/absorption} = \frac{\lambda \cancel{n}\cancel{dt}}{v\Sigma_a \cancel{n}\cancel{dt}} = \frac{\lambda}{v\Sigma_a}$$

$v = 2.2 \times 10^{5}$ cm/sec.

$\Sigma_a = 0.022\ \text{cm}^{-1}$

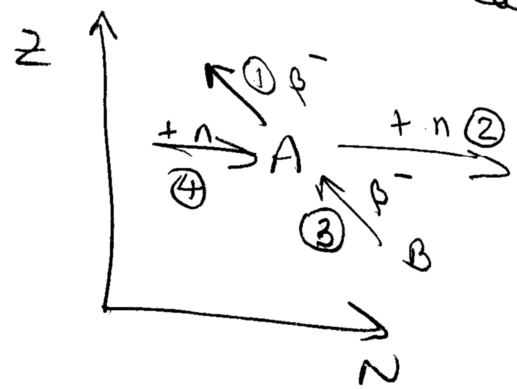
$$\lambda = \frac{\ln 2}{T_{\frac{1}{2}}} = \frac{0.693}{11.7 \times 60}\ \text{sec}^{-1} = 9.9 \times 10^{-4}\ \text{sec}^{-1}$$

$$\therefore\ \frac{\lambda}{v\Sigma_a} = 2.04 \times 10^{-7}$$, ie decay is not likely.

3)

a) $$\frac{dN_A}{dt} = -\underset{①}{\lambda_A N_A} - \underset{②}{\sigma_a^A \phi N_A} + \underset{③}{\lambda_B N_B} + \underset{④}{\sigma_\gamma^C \phi N_C}$$

(decay) (neutron capture) (decay of parent) (transmutation)



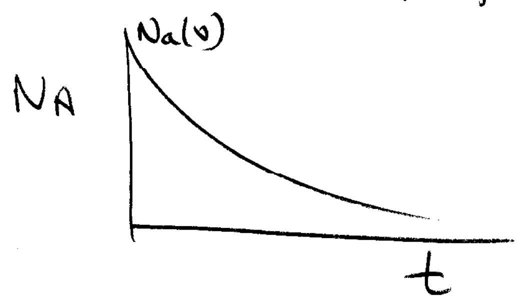
(could also have a fuel loading term $F(t)$)

b) If only capture:

$$\frac{dN_A}{dt} = -\sigma_a^A \phi N_A$$

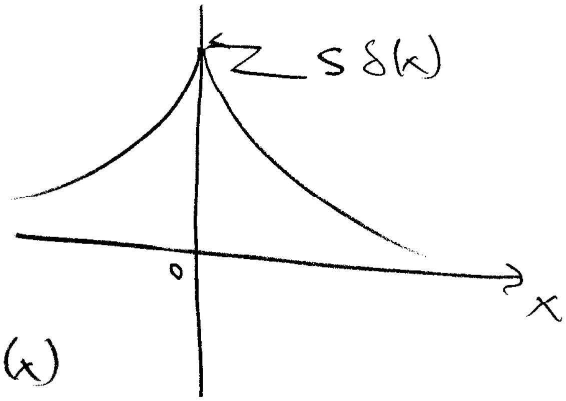
$$\therefore \quad N_A = N_A(0)\, e^{-\int_0^t \sigma_a^A \phi\, dt}$$

$$= N_A(0)\, e^{-\sigma_a^A \phi t} \quad \text{if } \phi = \text{constant.}$$



[Note: If you included ④ as a capture term, the solution is a bit more complex but doable if you know $N_C(t)$]

4.



a) $$\frac{1}{v}\underbrace{\cancel{\frac{\partial\phi}{\partial t}}}_{0} = D\frac{\partial^2\phi}{\partial x^2} - \Sigma_a\phi + S\delta(x)$$

$$\therefore\ -D\frac{\partial^2\phi}{\partial x^2} + \Sigma_a\phi = S\delta(x)$$

$$= 0 \text{ for } x \neq 0$$

Try $\phi = Ae^{-x/L} + Ce^{+x/L}$ where $L = D/\Sigma_a$

For $x > 0$

B.C: ① $\phi(\infty) = 0 \ \therefore\ C = 0$

② $J\big|_{x=0} = S/2 \simeq -D\frac{\partial\phi}{\partial x}\Big|_{x=0}$

$$\therefore\ \frac{DA}{L}e^{-x/L}\Big|_{x=0} = S/2$$

$$\therefore\ \frac{DA}{L} = S/2 \implies A = \frac{SL}{2D}$$

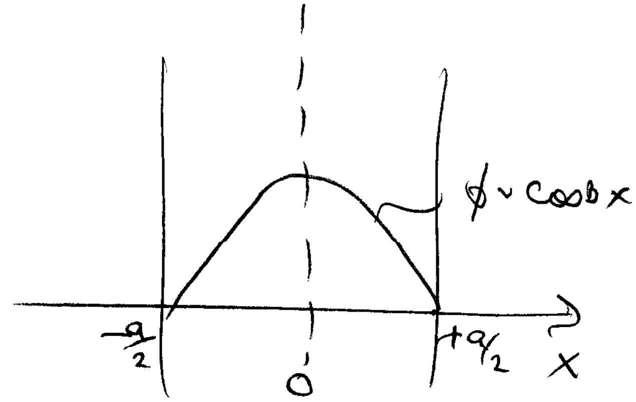
$$\therefore\ \phi = \frac{SL}{2D}e^{-x/L}.$$

Generalizing to $\pm x \Rightarrow \phi = \frac{SL}{2D}e^{-|x|/L}$

b) This is different than simple attenuation ($\phi = \phi_0 e^{-\Sigma x}$) since multiple scatters (diffusion) has been considered.

5. a)

$$D\frac{\partial^2\phi}{\partial x^2} + \left(-\Sigma_a + \nu\Sigma_f\right)\phi = 0$$



B.C.: 1. $\phi(+a/2) = 0$

2. $J\big|_{x=0} = 0 \Rightarrow \frac{\partial\phi}{\partial x}\Big|_{x=0} = 0$

or 1. $\phi(\pm a/2) = 0$

2. Symmetry

Try $\phi = \phi_0 \cos(Bx) \Rightarrow \phi = \phi_0 \cos(\pi/a\, x)$

BC 1 $\Rightarrow B = \pi/a$.

b) ∴ substituting back in to the differential eqn:

$$-DB^2 \cancel{\phi_0} + (\Sigma_a + \nu\Sigma_f)\cancel{\phi_0} = 0$$

$$\therefore -DB^2 - \Sigma_a + \nu\Sigma_f = 0$$

or $$\boxed{B_g^2 = (\pi/a)^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} \equiv B_m^2}$$

criticality condition.

This means that the losses (leakage + absorption) must equal the source (fission) precisely if the reactor is to be in steady state (ie to be critical).

#4. 2 group, homogeneous.

a) In steady state:

$$-D_1 \frac{\partial^2 \phi_1}{\partial x^2} \overbrace{+ \Sigma_{a_1}\phi_1 + \Sigma_{s_1}\phi_1 - \Sigma_{s_{11}}\phi_1}^{\equiv + \Sigma_{R1}\phi_1} - \Sigma_{s_{21}}\phi_2$$
$$-\cancel{\chi_1}\left(\nu_1 \Sigma_{f_1}\phi_1 + \nu_2 \Sigma_{f_2}\phi_2\right) = 0$$

$$-D_2 \frac{\partial^2 \phi_2}{\partial x^2} \overbrace{+ \Sigma_{a_2}\phi_2 + \Sigma_{s_2}\phi_2}^{\equiv + \Sigma_{R2}\phi_2} - \Sigma_{s_{12}}\phi_1 - \Sigma_{s_{22}}\phi_2$$
$$-\cancel{\chi_2\left(\nu_1 \Sigma_{f_1}\phi_1 + \nu_2 \Sigma_{f_2}\phi_2\right)} = 0$$

For bare reactor, ϕ_1 & ϕ_2 have same shape (cosine).
Usual B.C: $\phi(\pm a/2) = 0$ (at extrapolated distance)
Let $\phi_1(x) = \Phi_1 \psi(x)$, $\phi_2(x) = \Phi_2 \psi(x)$
$$+\nabla^2 \psi + B^2 \psi = 0$$

b) $\therefore$
$$\begin{bmatrix} D_1 B^2 + \Sigma_{R_1} - \chi_1 \nu_1 \Sigma_{f_1} & -\Sigma_{s_{21}} - \chi_1 \nu_2 \Sigma_{f_2} \\ -\Sigma_{s_{12}} \cancel{-\chi_2 \nu_1 \Sigma_{f_1}} & D_2 B^2 + \Sigma_{R2} \cancel{-\chi_2 \nu_2 \Sigma_{f_2}} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} = 0$$

ie $\underline{\underline{A}}\ \Phi = 0$

Which only has a non-trivial solution if $|\underline{\underline{A}}| = 0$

$$\therefore \boxed{\begin{aligned} &(D_1 B^2 + \Sigma_{R_1} - \cancel{\chi_1}\nu_1 \Sigma_{f_1})(D_2 B^2 + \Sigma_{R_2} \cancel{- \chi_2 \nu_2 \Sigma_{f_2}}) \\ &\quad - (\Sigma_{s_{21}} + \cancel{\chi_1}\nu_2 \Sigma_{f_2})(\Sigma_{s_{12}} \cancel{+ \chi_2 \nu_1 \Sigma_{f_1}}) = 0 \\ &\text{criticality condition} \end{aligned}}$$

Discussion:

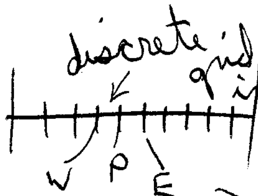
c) For 2 groups, no upscatter is likely. ie $\Sigma_{21} \simeq 0$ Also all fission neutrons are born in the fast group, ie $\chi_2 \simeq 0$, $\chi_1 = 1$. We can also ignore fast fissions usually, ie $\Sigma_{f_1} \simeq 0$ ($\sim 3\%$ are fast fissions actually. This is OK given the $\pm 5\%$ typical errors in measured Σ's.

With these assumptions:

$$(D_1 B^2 + \Sigma_{R_1})(D_2 B^2 + \underbrace{\Sigma_{R_2}}_{= \Sigma_{a_2} \text{ now}}) - \nu_2 \Sigma_{f_2} \Sigma_{S_{12}} = 0$$

is the criticality condition.

7. a) $-D_1 \frac{\partial^2 \phi_1}{\partial x^2} + \Sigma_{a1}\phi_1 + \Sigma_{s1}\phi_1 - \Sigma_{s11}\phi_1 - \Sigma_{s21}\phi_2 - \frac{\chi_1}{k}\left(\nu_1 \Sigma_{f1}\phi_1 + \nu_2 \Sigma_{f2}\phi_2\right) = 0$



$-D_2 \frac{\partial^2 \phi_2}{\partial x^2} + \Sigma_{a2}\phi_2 + \Sigma_{s2}\phi_2 - \Sigma_{s12}\phi_1 - \Sigma_{s22}\phi_2 - \frac{\chi_1}{k}\left(\nu_1 \Sigma_{f1}\phi_1 + \nu_2 \Sigma_{f2}\phi_2\right) = 0$

Approximate $D\frac{\partial^2 \phi}{\partial x^2}$ as $\frac{D}{\Delta x}\,\frac{(\phi_E - \phi_P) - D(\phi_P - \phi_W)}{\Delta x}$

$= \frac{D}{\Delta x^2}\left(\phi_W - 2\phi_P + \phi_E\right)$ (East, West and point in question).

& gather up terms:

$\left(\Sigma_{a1} + \Sigma_{s1} - \Sigma_{s21} + \frac{2D}{\Delta x^2}\right)\phi_{1P}$

$= \left[\frac{D}{\Delta x^2}\phi_{1W} + \frac{D}{\Delta x^2}\phi_{1E} + \frac{\overset{=1.0}{\chi_1}}{k}\underbrace{\left(\nu_1 \Sigma_{f1}\phi_{1P} + \nu_2 \Sigma_{f2}\phi_{2P}\right)}_{A}\right]$

& $\left(\Sigma_{a2} + \Sigma_{s2} - \Sigma_{s22} + \frac{2D}{\Delta x^2}\right)\phi_{2P}$

$= \left[\frac{D}{\Delta x^2}\phi_{2W} + \frac{D}{\Delta x^2}\phi_{2E} + \Sigma_{s12}\phi_1 + \frac{\overset{=0.0}{\chi_2}}{k}\left(\;\cdots\;\right)\phi_{1P}\right]$

b) Algorithm
(0) Guess ϕ_1 & ϕ_2 distribution
(1) Sweep space, solving for ϕ_{1P} & ϕ_{2P}
(2) Update k: $k_{new} = k_{old} \times \frac{\Sigma A^{new}}{\Sigma A^{old}}$.
(3) Go to (1) & loop until converged.

c) All 3 models (numerical 2 group, analytical one group and analytical 2 group) are but different approximations to reality - crude ones at that.

Analytically, the criticality condition is a constraint on ~~size~~ + Σ_a, $\nu\Sigma_f$ etc.

We could have written

$$B_g^2 = (\pi/a)^2 = \frac{\frac{\nu\Sigma_f}{K} - \Sigma_a}{D} = B_m^2$$

where the K is analogous to the fudge factor K used in the numerical method.

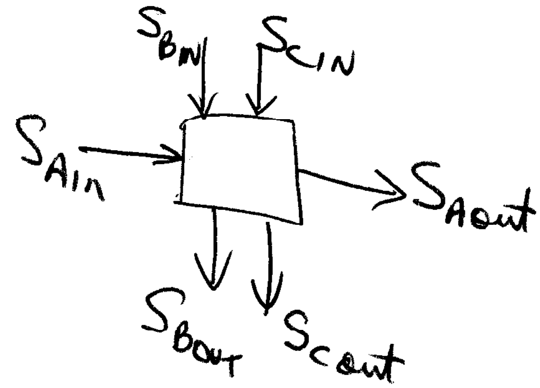
Each method just gives a different approximation to the fudge factor needed to adjust the model to criticality. This is a measure of what the model (with given size + materials) predicts re criticality.

To get a better estimate, use more groups + more grid points.

8.

a) $A(n,\gamma)B$

ie $A + n \rightarrow B + \gamma$



$$\frac{\partial N_A}{\partial t} = S_{A_{IN}} - S_{A_{OUT}} - \sigma_A N_A \phi$$

$$\frac{\partial N_B}{\partial t} = S_{B_{IN}} - S_{B_{out}} + \sigma_A N_A \phi - \sigma_B N_B \phi - \lambda_B N_B \qquad \lambda_B = \frac{\ln 2}{T_{1/2}}$$

$$\frac{\partial N_C}{\partial t} = S_{C_{IN}} - S_{C_{out}} - \sigma_C N_C \phi + \lambda_B N_B$$

N, S, ϕ are $f_n(t)$

b) $N_A^{t+\Delta t} = N_A^t + \Delta t\,(S_{A_{IN}} - S_{A_{out}} - \sigma_A \phi N_A)$

N_A^t for explicit, $N_A^{t+\Delta t}$ for implicit

Using semi-implicit:

$$N_A^{t+\Delta t} = \frac{N_A^t + \Delta t\,(S_{A_{IN}} - S_{A_{out}})}{(1 + \Delta t\, \sigma_A \phi)}$$

$$N_B^{t+\Delta t} = \frac{N_B^t + \Delta t\,(S_{B_{IN}} - S_{B_{out}} + \sigma_A N_A^t \phi)}{(1 + \Delta t\,(\sigma_B \phi + \lambda_B))}$$

$$N_C^{t+\Delta t} = \frac{N_C^t + \Delta t\,(S_{C_{in}} - S_{C_{out}} + \lambda_B N_B)}{1 + \Delta t\, \sigma_C \phi}$$

start (initial N_A, N_B, N_C)

↓

loop $N_A^{t+\Delta t} = \ldots$

$N_B^{t+\Delta t} = \ldots$

$N_C^{t+\Delta t} = \ldots$

$t > t_{finish}$?

No → loop

Yes → stop

9

a) $(\Sigma_{f_1}\phi_1 + \Sigma_{f_2}\phi_2)$ $\quad$ $(\sigma_{a_1}^X\phi_1 + \sigma_{a_2}^X\phi_2)$

$$\frac{\partial X}{\partial t} = \gamma_X \Sigma_f \phi + \lambda_I I - \lambda_X X + \sigma_a^X \phi X, \quad \frac{\partial I}{\partial t} = \gamma_I \Sigma_f \phi - \lambda_I I$$

Where $\phi = \phi_0 \cos Bx$ as usual. $B = \pi/a$.

In steady state: $I = \dfrac{\gamma_I \Sigma_f \phi_0}{\lambda_I} \cos Bx$

$\therefore$ for Xe: $0 = \gamma_X \Sigma_f \phi + \gamma_I \Sigma_f \phi - \lambda_X X - \sigma_a^X \phi X$

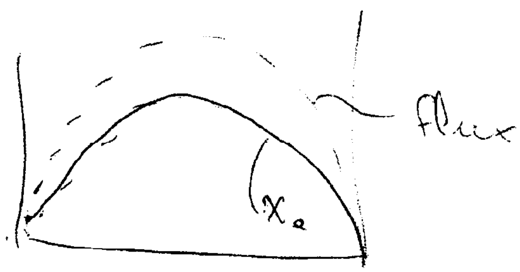
$$\therefore \boxed{X = \frac{(\gamma_X + \gamma_I)\Sigma_f \phi_0 \cos Bx}{\lambda_X + \sigma_a^X \phi_0 \cos Bx}}$$

It is not a simple cosine.

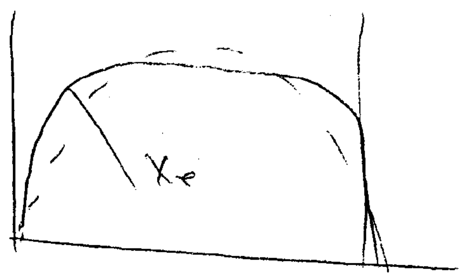
mostly ϕ_2

b) low ϕ_0

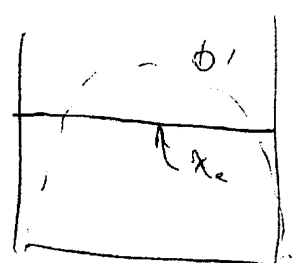
$$X = \frac{(\gamma_X + \gamma_I)}{\lambda_X} \Sigma_a \phi_0 \cos Bx$$



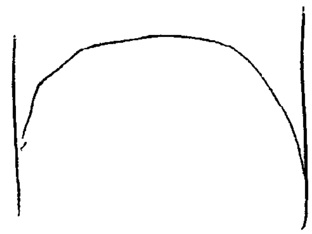
med



high $X = \dfrac{(\quad)\Sigma_f \cancel{\phi_0 \cos Bx}}{\sigma_a^X \cancel{\phi_0 \cos Bx}}$



(Note: This is only approximate since Xe will perturb the flux).



$$\frac{1}{v_1}\frac{\partial \phi_1}{\partial t} = 0 = D_1 \frac{\partial^2 \phi_1}{\partial x^2} - \Sigma_a \phi_1 - \Sigma_{s_1} \phi_1 + \nu_2 \Sigma_{f_2} \phi_2$$

$$\frac{1}{v_2}\frac{\partial \phi_2}{\partial t} = 0 = D_2 \frac{\partial^2 \phi_2}{\partial x^2} - \Sigma_a \phi_2 + \Sigma_{s_{12}} \phi_1$$

$\Sigma_{f_2} \phi_2$ = fissioning rate which leads to Xe & I.

$\therefore$ appropriate $\phi = \phi_2$ (also have minor term $\Sigma_{f_1} \phi_1$)

but spatial dist^n is $\phi_1 \sim \cos \frac{\pi x}{a}$

$\sim \phi_2$.

10 $\rho C_p \frac{\partial T}{\partial t} = q''' \leftarrow$ heat generation rate

$\therefore \quad \rho C_p \Delta T \simeq q''' \Delta t$

$\therefore \quad \underbrace{\rho V}_{\text{Mass}} C_p \Delta T \simeq \underbrace{q''' V \Delta t}_{\text{total heat generated}} = Q \quad , \; V = \text{volume}.$

For a mixture of Al & U, the ΔT is the same for both & the Q is distributed into the combined homogeneous mass, ie:

$$\left(M_{Al} C_{p_{Al}} + M_u \, C_{p_u} \right) \Delta T = Q$$

$$\therefore \quad \Delta T = \frac{Q}{M_{Al} C_{p_{Al}} + M_u C_{p_u}}$$

$$= \frac{2.2 \times 10^6 \text{ J}}{(24.76 \times 903.5 + 7.539 \times 201.6) \text{ J}/°\text{K}}$$

$$= 92.09 \; °\text{K}$$