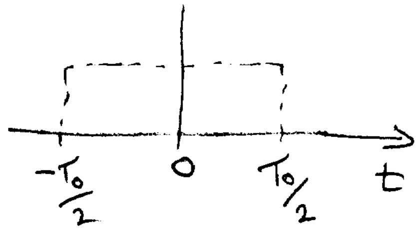
9.1 The Hanning Window is defined as:

$$f(t) = 1 + \cos\left(\frac{2\pi t}{T_0}\right) = \cos^2\left(\frac{\pi t}{T_0}\right)$$

$$\text{for } |t| \leq T_0$$

$$= 0 \quad , \quad |t| > T_0$$

where T_0 is the window size



(a) Show that the Fourier Transform of the Hanning Window is:

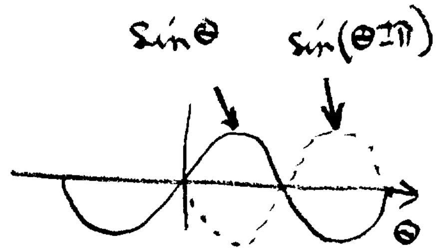
$$F(\nu) = \frac{T_0 \sin(\pi \nu T_0)}{\pi \nu T_0} + \frac{T_0 \sin(\pi(1-\nu T_0))}{2\pi(1-\nu T_0)} + \frac{T_0 \sin(\pi(1+\nu T_0))}{2\pi(1+\nu T_0)}$$

Hint: Recall: $\cos\theta = \dfrac{e^{i\theta} + e^{i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

(b) Show that $F(\nu)$ above reduces to:

$$F(\nu) = \frac{T_0 \sin \pi \nu T_0}{\pi \nu T_0 (1 - (\nu T_0)^2)}$$

Hint: $\sin\theta = -\sin(\theta + \pi)$
$= -\sin(\theta - \pi)$



(c) Sketch $f(t)$ + $F(\nu)$.

Sol'n:

(a) $$F(\nu) = \int_{-\infty}^{\infty} f(t)\, e^{-2\pi i \nu t} dt = \int_{-T_0/2}^{T_0/2} \left(1 + \cos \frac{2\pi t}{T_0}\right) e^{-2\pi i \nu t} dt$$

$$= \underbrace{\int_{-T_0/2}^{T_0/2} e^{-2\pi i \nu t} dt}_{1} + \underbrace{\int_{-T_0/2}^{T_0/2} \cos \frac{2\pi t}{T_0}\, e^{-2\pi i \nu t} dt}_{2}$$

$$1 = \frac{1}{(-2\pi i \nu)} e^{-2\pi i \nu t}\Big|_{-T_0/2}^{T_0/2} = \frac{e^{-2\pi i \nu T_0/2} - e^{2\pi i \nu T_0/2}}{-\pi\nu} \cdot \frac{T_0}{T_0} = \frac{T_0 \sin(\pi \nu T_0)}{\pi \nu T_0}$$

$$2 = \int_{-T_0/2}^{T_0/2} \left(\frac{e^{2\pi i T_0/2} + e^{-2\pi i T_0/2}}{2}\right) e^{-2\pi i \nu t} dt = \int_{-T_0/2}^{T_0/2} \left[\frac{e^{2\pi i t(1/T_0 - \nu)} + e^{-2\pi i t(1/T_0 + \nu)}}{2}\right] dt$$

$$= \frac{1}{2} \frac{e^{2\pi i t(1/T_0 - \nu)}}{2\pi i (1/T_0 - \nu)}\Bigg|_{-T_0/2}^{T_0/2} + \frac{1}{2} \frac{e^{-2\pi i t(1/T_0 + \nu)}}{-2\pi i (1/T_0 + \nu)}\Bigg|_{-T_0/2}^{T_0/2}$$

$$= \frac{1}{2}\left[\frac{e^{2\pi i \frac{T_0}{2}(\frac{1}{T_0} - \nu)} - e^{2\pi i \frac{T_0}{2}(1/T_0 - \nu)}}{2\pi i (1/T_0 - \nu)}\right] + \frac{1}{2}\left[\frac{e^{-2\pi i \frac{T_0}{2}(1/T_0 + \nu)} - e^{2\pi i \frac{T_0}{2}(\frac{1}{T_0} + \nu)}}{-2\pi i (1/T_0 + \nu)}\right]$$

$$= \frac{T_0}{2} \frac{\sin \pi (1 - \nu T_0)}{\pi (1 - \nu T_0)} + \frac{T_0}{2} \frac{\sin \pi (1 + \nu T_0)}{\pi (1 + \nu T_0)}$$

$$\therefore F(\nu) = 1 + 2$$

$$= \frac{T_0 \sin(\pi \nu T_0)}{\pi \nu T_0} + \frac{T_0 \sin \pi (1 - \nu T_0)}{2\pi (1 - \nu T_0)} + \frac{T_0 \sin \pi (1 + \nu T_0)}{2\pi (1 + \nu T_0)}$$

QED

(b) We have:

$$F(\nu) = \frac{T_0 \sin(\pi \nu T_0)}{\pi \nu T_0} + \frac{T_0 \sin(\pi(1-\nu T_0))}{2\pi(1-\nu T_0)} + \frac{T_0 \sin(\pi(1+\nu T_0))}{2\pi(1+\nu T_0)}$$

Since $\sin x = -\sin(-x)$

$$F(\nu) = \frac{T_0 \sin(\pi \nu T_0)}{\pi \nu T_0} - \frac{T_0 \sin(\pi \nu T_0 - \pi)}{2\pi(1-\nu T_0)} + \frac{T_0 \sin(\pi \nu T_0 + \pi)}{2\pi(1+\nu T_0)}$$

$$= \frac{T_0 \sin(\pi \nu T_0)}{\pi \nu T_0} + \frac{T_0 \sin(\pi \nu T_0)}{2\pi(1-\nu T_0)} - \frac{T_0 \sin(\pi \nu T_0)}{2\pi(1+\nu T_0)}$$

since $\sin\theta = -\sin(\theta \pm \pi)$

$$= \frac{T_0 \sin(\pi \nu T_0)}{\pi}\left[\frac{1}{\nu T_0} + \frac{1}{2(1-\nu T_0)} - \frac{1}{2(1+\nu T_0)}\right]$$

$$= \frac{T_0}{\pi}\sin(\pi \nu T_0)\left[\frac{1}{\nu T_0} + \frac{\nu T_0}{2[1-(\nu T_0)^2]}\right]$$

$$= \frac{T_0}{\pi}\sin(\pi \nu T_0)\left[\frac{1}{\nu T_0[1-(\nu T_0)^2]}\right]$$

$$\therefore F(\nu) = \frac{T_0 \sin(\nu \pi T_0)}{\pi \nu T_0 [1-(\nu T_0)^2]}$$

QED

9.2 When would you use a Hanning window and when would you use a rectangular window?

Sol'n

If your signal has a significant non-zero value when you start and/or stop the signal sampling (ie, if $s(t)$ is significant at the window edges) then $S(\nu)$ will have frequency artifacts. Therefore, use the Hanning (or similar) window to bring $s(t) \Rightarrow 0$ at and near the window edges. But this distorts the signal skirt as it reduces the unwanted ripples.

Use a rectangular window when the signal is completely contained within the window. Use a non-rectangular window only if you have to.

9.3 What is the F.T. of: $\Pi_4(t)\cos 2\pi t$

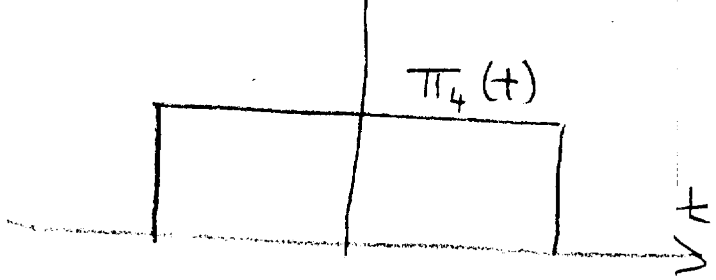
Sketch the signal in both the t & ν domains.

(This is a cosine that has been chopped by a rectangular window)

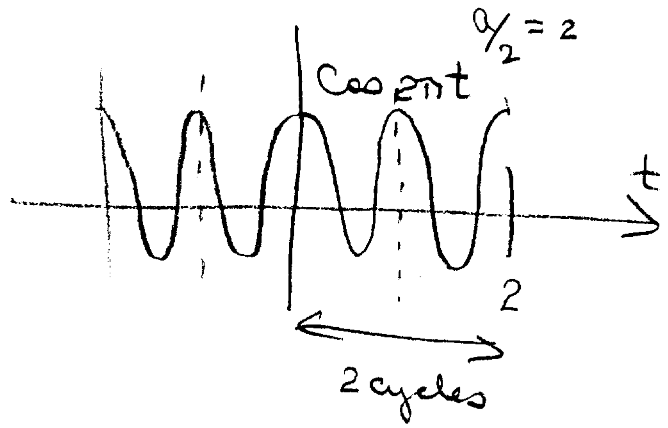
Sol'n

In general: The signal is $\Pi_a(t)\cos 2\pi b t$.

Here $a = 4$, $b = 1$



$\rightleftharpoons \quad 4\,\mathrm{sinc}(4\pi\nu)$



$\rightleftharpoons \quad \frac{1}{2}\left[\delta(\nu-1)+\delta(\nu+1)\right]$

Now: $f(t)\,g(t) \rightleftharpoons F(\nu) * G(\nu)$

$\therefore\ \Pi_4(t)\cos 2\pi t \rightleftharpoons 4\,\mathrm{sinc}(4\pi\nu) * \frac{1}{2}\left[\delta(\nu-1)+\delta(\nu+1)\right]$

$\rightleftharpoons \quad 2\,\mathrm{sinc}(4\pi(\nu-1)) + 2\,\mathrm{sinc}(4\pi(\nu+1))$