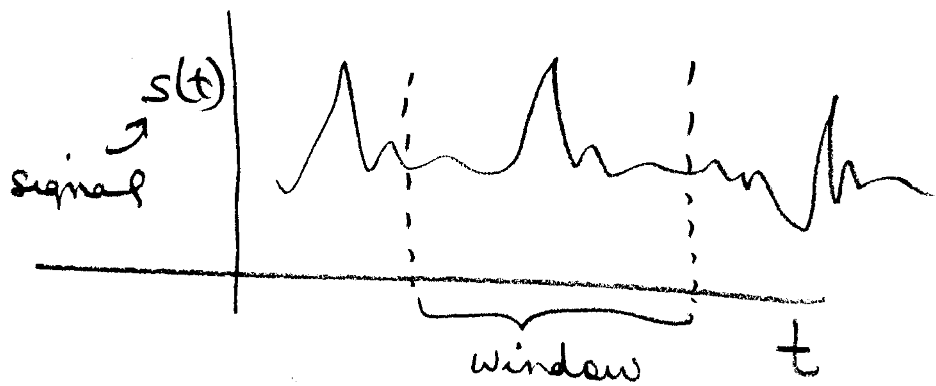
Chapter 9 Windows & Discrete FT

(following Fante 3.5)

9.1 Windows

Whereas filters had to do with frequency space, windows refer to time ~~space~~.

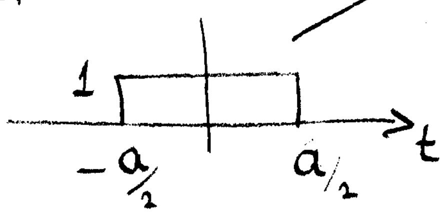
Windows in time occur invariably and naturally: our measurements must start and stop somewhere.



g(t)

Let g (t) be the window. Then:

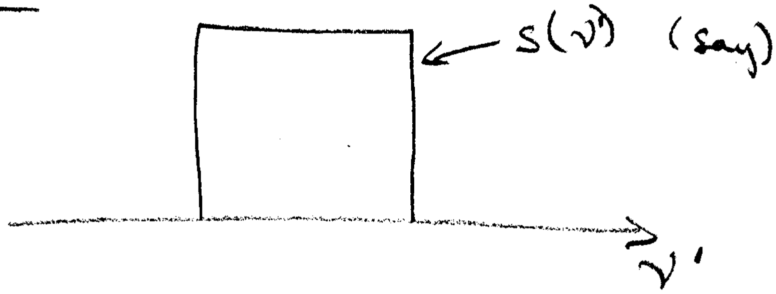
$$\underset{\text{windowed signal}}{w(t)} = s(t)\underset{\Pi_a(t)}{g(t)} \rightleftharpoons S(\nu) * \underset{\mathrm{sinc}(\pi\nu a)}{G(\nu)} \equiv W(\nu)$$

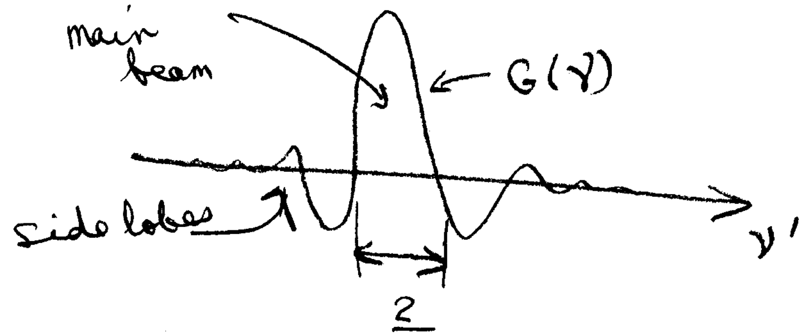


$$= \int_{-\infty}^{\infty} S(\nu')\, G(\nu-\nu')\, d\nu'$$

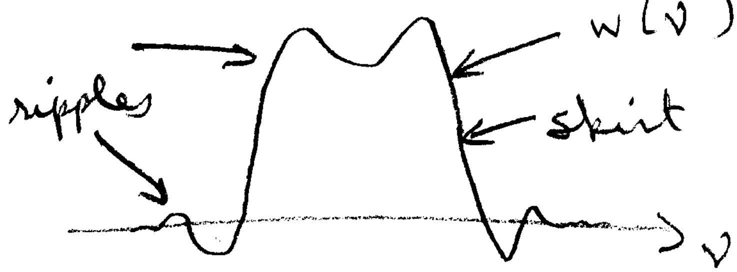
$$= \int_{-\infty}^{\infty} S(\nu')\, \mathrm{sinc}\left(\pi a(\nu-\nu')\right) d\nu'$$

Example





after convolution

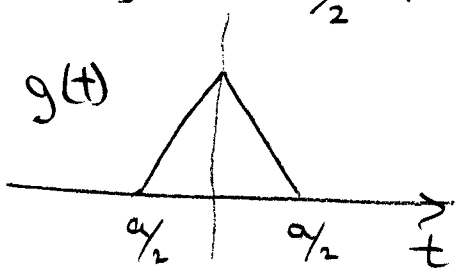


Windowing corrupts the signal. In effect, the Fourier Transform is built on the premise ~~that~~ $s(t)$ is 0 at $\pm\infty$. Since the F.T. assumes limits of $\pm\infty$, the window effectively says that the signal drops to 0 at the window edges and stays 0 from then on. If the signal were finite at the window edge, then it experiences a false truncation. This introduces false oscillations in the frequency domain (ripples) and a false 'skirt'.

↑ caused by the side lobes ← caused by the main beam of $\text{sinc}(\pi\nu(f-f'))$

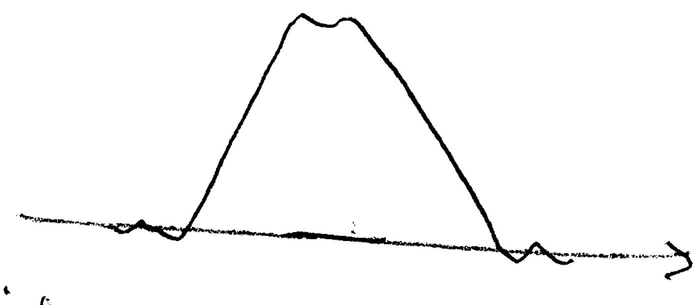
Let's try another window shape to see if we can do better. Let's try $\Lambda_a(t)$, the triangular window. This won't start and stop so abruptly so the ripples should be less but it will cause the loss of signal information near the window edges.

Now, $\Lambda_{a/2}(t) \rightleftharpoons \left(\frac{a}{2}\right)^2 \operatorname{sinc}^2(\pi \nu a/2)$



Thus:

W(ν) is



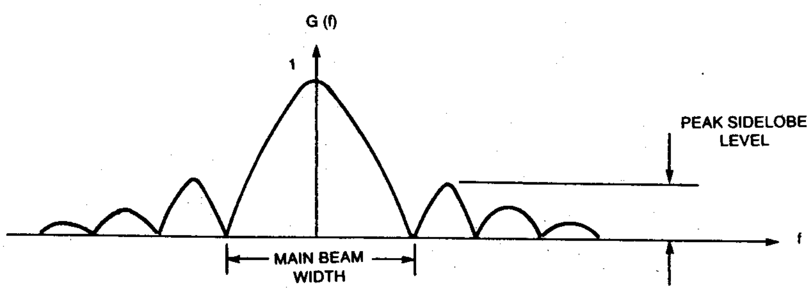
Note: the ripples are smaller but the skirt is bigger (more tapered)

In general, the ripples can be reduced (by making $g=0$ at edges), but at the expense of a broader skirt (due to loss of info near the edges).

A number of common windows are as follows:

From Fante pages 84-85

TABLE 3.4 Properties of Some Commonly Used Windowing Functions



Window $g(t)$	Fourier Transform $G(f)$: Width of Main Beam (between Zeros)	Fourier Transform $G(f)$: Peak Sidelobe Amplitude (Relative to Main Beam Peak)
$g(t)=1,\ \lvert t\rvert \le \frac{T_o}{2}$; $=0,\ \lvert t\rvert > \frac{T_o}{2}$ (rectangular)	$\frac{2}{T_o}$	0.22
$g(t)=1-\frac{2\lvert t\rvert}{T_o},\ \lvert t\rvert \le \frac{T_o}{2}$; $=0,\ \lvert t\rvert > \frac{T_o}{2}$ (Bartlett)	$\frac{4}{T_o}$	0.056
$g(t)=\cos^2\left(\frac{\pi t}{T_o}\right),\ \lvert t\rvert \le \frac{T_o}{2}$; $=0,\ \lvert t\rvert > \frac{T_o}{2}$ (Hanning)	$\frac{4}{T_o}$	0.028
$g(t)=0.54+0.46\cos\frac{2\pi t}{T_o},\ \lvert t\rvert \le \frac{T_o}{2}$; $=0,\ \lvert t\rvert > \frac{T_o}{2}$ (Hamming)	$\frac{4}{T_o}$	0.0089
$g(t)=0.42+0.5\cos\frac{2\pi t}{T_o}+0.08\cos\frac{4\pi t}{T_o},\ \lvert t\rvert \le \frac{T_o}{2}$; $=0,\ \lvert t\rvert > \frac{T_o}{2}$ (Blackman)	$\frac{6}{T_o}$	0.0014
$g(t)=\frac{I_0[\pi\beta\sqrt{1-(2t/T_o)^2}]}{I_0(\pi\beta)},\ \lvert t\rvert \le \frac{T_o}{2}$ (Taylor)	$\frac{2(\beta^2+1)^{1/2}}{T_o}$	$0.22\left(\frac{\sinh \pi\beta}{\pi\beta}\right)^{-1}$

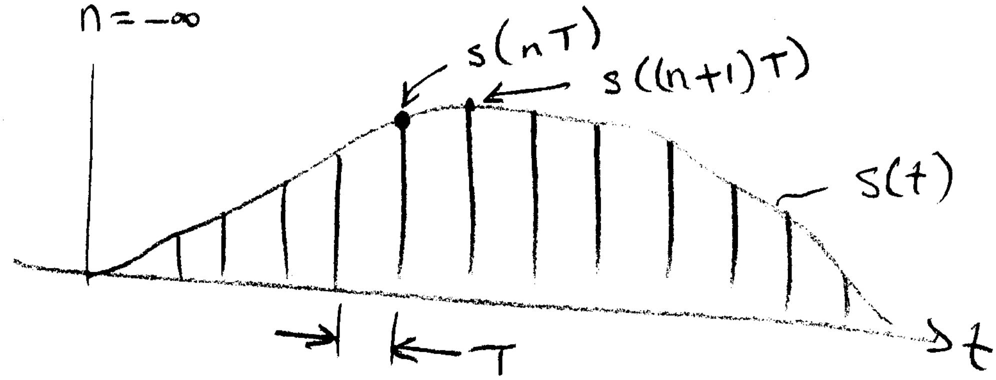
I_0 = modified Bessel function

9.2 Discrete Fourier Transforms (following Fante chapter 5)

There is a whole parallel development for the discrete (ie sampled) signal as for the continuous signal. We have already met the sampled signal in our discussion on aliasing in chapter 6.

Recall ~~that~~ we ~~showed that~~

$$\hat{S}(\nu) = \sum_{n=-\infty}^{\infty} s(nT)\, e^{-2\pi i \nu n T}$$



We ~~showed~~ how the FT. of $s(nT)$ is periodic and ~~that~~ $\hat{S}(\nu) = \frac{1}{T} \sum_{n=-\infty}^{\infty} S(\nu + n/T)$ so ~~that~~

we need to limit ν or increase the sampling rate $1/T$ if we are to capture $S(\nu)$ faithfully. If B is the ~~bandwidth~~ of $S(\nu)$, then we must respect: $\frac{1}{T} \geq B$ (Nyquist criteria)

But what if we want to reconstruct $s(t)$ from $\hat{S}(\nu)$. We will have $\hat{S}(\nu)$ in some discrete form in a computer. So in doing the inverse FT of a discrete signal, $\hat{S}(\nu)$, we meet the same situation as we found when we tried to do the F.T. of $s(nT)$ - only this time the roles of ν and t are reversed.

So we expect the same aliasing problem - the reconstructed $s(t)$ will be replicated in an erroneous periodic fashion.

The criteria $\frac{1}{T} \geq B$ translates to

$$\frac{1}{\Delta\nu} \geq T_0 \quad \text{ie} \quad \Delta\nu \geq \frac{1}{T_0}$$

Note: T = sampling rate $\Rightarrow \Delta\nu$
B = bandwidth. $\Rightarrow$ time window, T_0

or $\Delta\nu = \frac{1}{T_0}$ at the critical sampling rate.

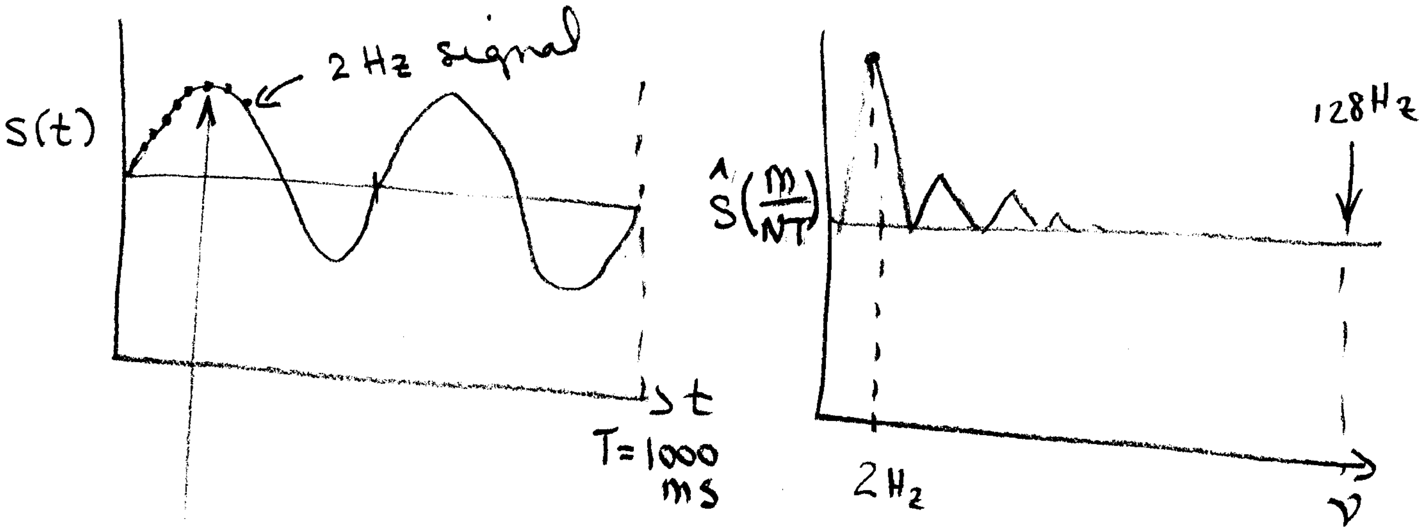
So if we had sampled for 1 second, the frequency resolution is

$$\Delta\nu = 1\text{ Hz}.$$

Thus rather then working with $\hat{S}(\nu)$ we deal with $\hat{S}$ in steps of $\Delta\nu$, ie

$$\hat{S}(m\Delta\nu) \quad \text{or} \quad \hat{S}\left(\frac{m}{T_0}\right) = \hat{S}\left(\frac{m}{NT}\right)$$

So you will see in commercial code the following displays:



Samples, say 256 samples for the 1000 ms window

$\Rightarrow T = \frac{1}{256}$ s sampling rate

$\therefore$ highest frequency (Nyquist limit)

is $\frac{B}{2} = \frac{1}{2T} = \frac{1}{2 \times \frac{1}{256}} = \frac{256}{2} = 128$ Hz

One final point, you will also meet 'zero padding' in commercial code. Zero padding simply adds zeros at the end of the time signal.

Let's double the signal length from N samples to $2N$ samples. Then we have

$$\hat{S}\left(\frac{m}{2NT}\right) = \sum_{n=0}^{2N-1} s(nT)\, e^{-2\pi i \frac{m}{2NT} nT}$$

this is a windowed case

$$= \sum_{n=0}^{N-1} s(nT)\, e^{-2\pi i (m/2)(n/N)}$$

Compare this to the original:

$$\hat{S}\left(\frac{m}{NT}\right) = \sum_{n=0}^{N-1} s(nT)\, e^{-2\pi i \frac{mn}{N}}$$

The only difference is $m \to m/2$, ie we interpolate the spectrum.

So zero padding is used to interpolate a signal to get a smoother appearance – but be warned: It does not increase the resolution since that is limited by $\Delta\nu = \frac{1}{T_0}$.

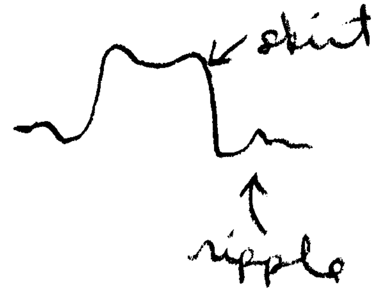
9.3 Recap

Window $= g(t)$, signal $= s(t)$

$$w = s \cdot t \quad \rightleftharpoons \quad S(\nu) * G(\nu) \equiv W(\nu)$$

$$= \int_{-\infty}^{\infty} S(\nu')\, G(\nu - \nu')\, d\nu'$$

If $S(\nu) = \Pi \Rightarrow W(\nu) =$



Try to choose $g(t)$ to reduce ripple without creating to much skirt.

DFT

$$\hat{S}\left(\frac{m}{NT}\right) = \sum_{n=-\infty}^{\infty}{}' \, s(nT)\, e^{-2\pi i \frac{mn}{N}}$$

$\frac{1}{T} > B$ + $\frac{1}{\Delta\nu} \geq T_0$ (frequency resolution limit)

aliasing limit

Use zero padding to get smoother appearance.

— The End —