Chapter 8 Application: Signal Analysis and Filters

(following James, Chapter 4)

8.1 Communication Channels

Let's take some signal, say a voltage, $V(t)$.

The average signal is just

$$\langle V(t) \rangle = \frac{1}{2T} \int_{-T}^{T} V(t)\,dt$$

denotes an average

note we are reflecting the signal about the origin to keep it an even function. That way the F.T. is real.

The power through a unit resistor is:

$$\langle V^2(t) \rangle = \frac{1}{2T} \int_{-T}^{T} V^2(t)\,dt$$

For real signals $V(t)$ has a finite duration so we can let $T \to \infty$ without worrying that the $\int$ becomes unbounded.

We define $V(t) \rightleftharpoons C(\nu)$ & recall:

$$\int_{-\infty}^{\infty} V^2(t)\,dt = \int_{-\infty}^{\infty} \underbrace{|C(\nu)|^2}_{\equiv\, G(\nu)}\,d\nu$$

Rayleigh's Theorem.

Spectral Power Density

Note: James' definition differs by a constant of proportionality.

We can relate the SPD to the autocorrelation function as follows:

$$A(\tau) \equiv \text{autocorrelation}$$
$$= \int_{-\infty}^{\infty} v(t)\, v(t+\tau)\, dt$$

We showed in chapter 6 that

$$A(\tau) \rightleftharpoons C^*(\nu)\, C(\nu)$$

Since $C^*(\nu)\, C(\nu) = |C(\nu)|^2$

$$\equiv G(\nu),$$

we state, in words:

"The spectral power density is the F.T. of the autocorrelation function of the signal."

This is the Wiener - Khinchine Theorem.

8.2 Noise

Types:

White noise — random, ie equal power density at all frequencies

$A(\tau) = 0$ since random, ie no correlation in the noise, except at $\tau = 0$. $A(\tau) \propto \delta(\tau)$.

Electron shot noise / Johnson noise — fluctuations of voltage in a resistor

$$\langle V^2(t) \rangle = 4 R k T \Delta\nu$$

- $\Delta\nu$: bandwidth of signal
- T: absolute temp.
- k: Boltzmann's const
- R: Resistance

$$\therefore V_{RMS} = 1.3 \times 10^{-10} (R \Delta\nu)^{\frac{1}{2}} \text{ volts}$$

Photon shot noise — noise due to individual photons. Not normally an issue since a typical laser gives 10^{18} photons per second. At 100 MHz sampling, that is 10^{10} photons/sample.

Semi-conductor noise —

SPD varies as $1/\nu$. So semi-conductor detectors work better at higher frequencies (but not so high as to get significant shot noise due to short sample period).

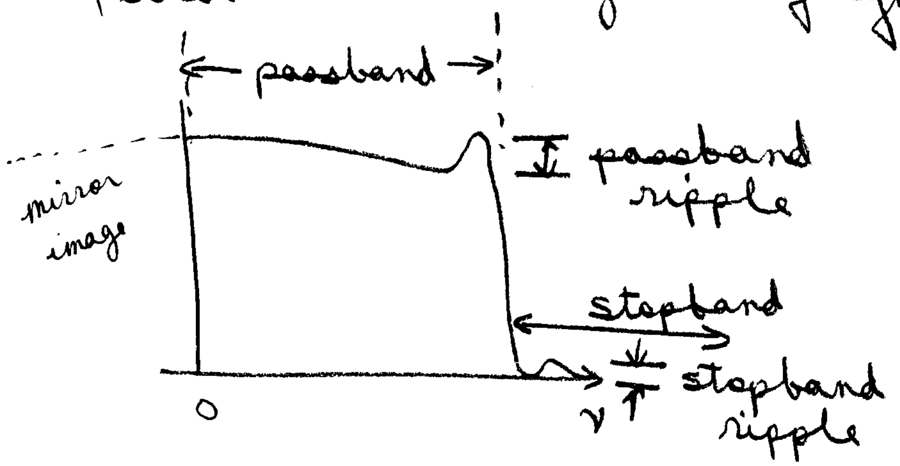
8.3 Filters

(Loosely based on James, 4.3 plus www.bores.com)

A filter is some device (eg, electronics or software) that alters the amplitude and phase of a signal. Thus,

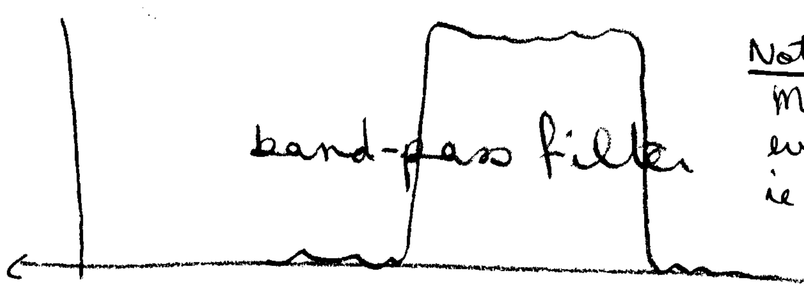
$$V_{out} = Z(\nu)\, V_{in} = A(\nu)\, e^{i\phi(\nu)}\, V_{in}\,.$$

When we speak of filters, we are referring to a device ~~that~~ alters the frequency. For example, a low-pass filter lets low frequencies through but attenuates high frequencies. A high-pass filter does the opposite and a band-pass filter lets through a frequency range.



low-pass filter

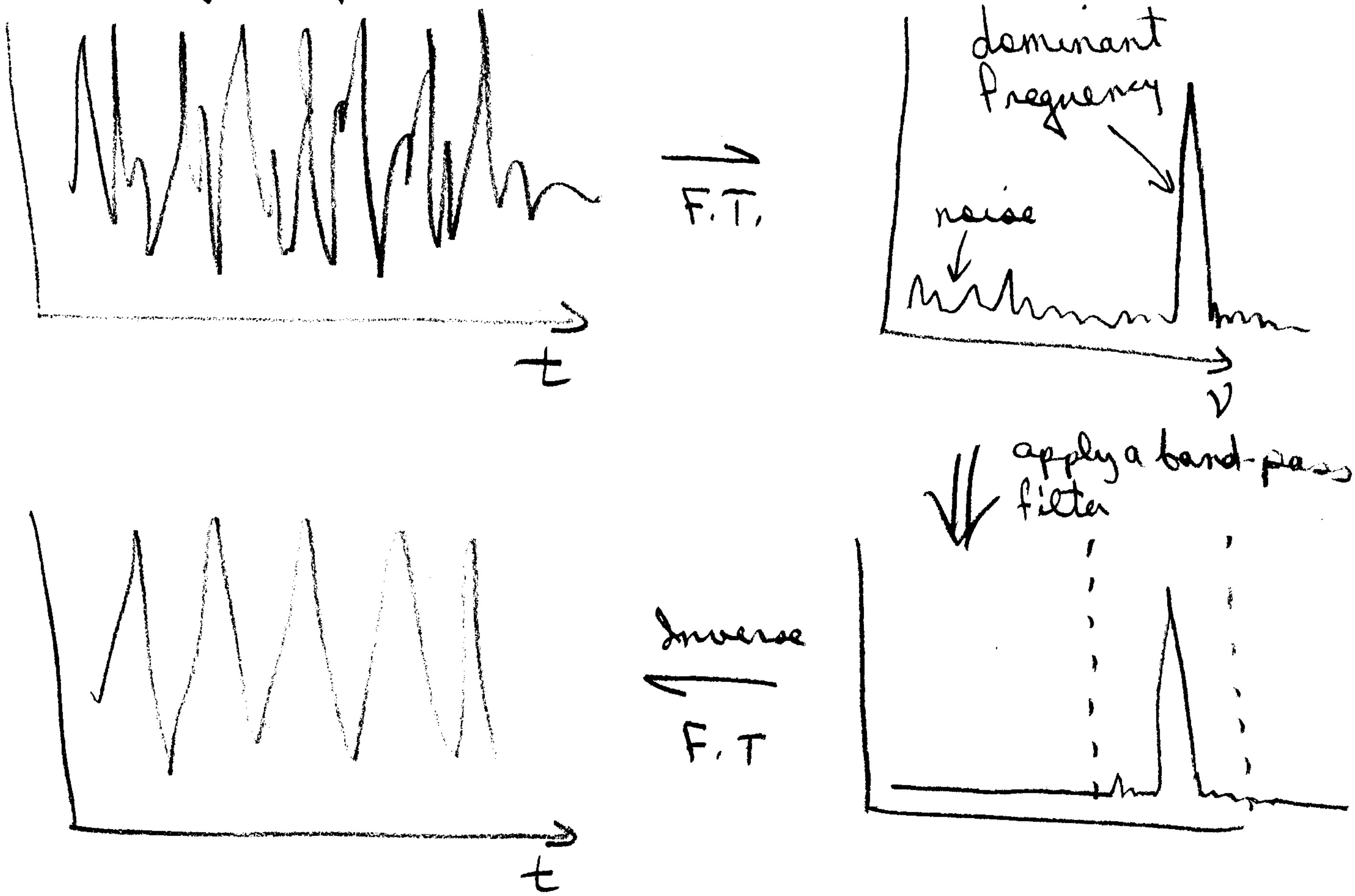
high-pass filter



band-pass filter

Note: Make all filters even functions, ie 'mirror' them.

You can use filtering to clean up a noisy signal:



clean signal.

By filtering, you remove the unwanted frequencies.

Obviously, unless it is clear what is noise and what is not, you can generate garbage this way. Buyer beware.

Filtering is an art as much as it is a science and we won't get into it here, except for a few observations...

Your measuring device will have limits on its bandwidth, so, like it or not, your signal will have been filtered to some extent. You should always be aware of the devices limitations when making a measurement.

As we shall see, filters can be good or bad - they can get rid of unwanted noise but they can also introduce unreal wiggles in your signal.

8.4 The Matched Filter Theorem (James 4.4)

What filter should be used to get the best signal to noise ratio?

Consider a signal $V_i(t)$ with F.T. of $C(\nu)$ + SPD of $S(\nu) = |C(\nu)|^2$.

Apply a filter $Z(\nu)$ to give an output whose output in the frequency domain is $C(\nu)Z(\nu)$.

Thus the output SPD is $|C(\nu)Z(\nu)|^2$.

Let's say there is noise in the input signal with a SPD of $|N(\nu)|^2$.

Therefore, the total signal power $= \int_{-\infty}^{\infty} |C(\nu)Z(\nu)|^2 d\nu$

and the noise power is $\int_{-\infty}^{\infty} |N(\nu)Z(\nu)|^2 d\nu$.

For white noise $|N(\nu)|^2 = A =$ constant.

$$\therefore \frac{\text{Signal}}{\text{Noise}} = \frac{\int_{-\infty}^{\infty} |C(\nu)Z(\nu)|^2 d\nu}{A \int_{-\infty}^{\infty} |Z(\nu)|^2 d\nu}$$

Schwartz's inequality states:

$$\left[\int_{-\infty}^{\infty} |C(\nu)Z(\nu)|^2 d\nu\right]^2 \leq \int_{-\infty}^{\infty} |C(\nu)|^2 d\nu \int_{-\infty}^{\infty} |Z(\nu)|^2 d\nu$$

$$\therefore \frac{S}{N} \leq A \int_{-\infty}^{\infty} |C(\nu)|^2 d\nu$$

where the equality holds iff <u>$C(\nu)$ is a multiple of $Z(\nu)$</u>.

Aside

You can intuitively see this by thinking of the integral of $C(\nu) Z(\nu)$. It will be maximized when C & Z have the same form. If C was +ve when Z was −ve, then you'd get a negative contribution to the integral. But if both were +ve or both were −ve, you'd get positive contributions.

At any rate, from all this we conclude:

"The $\frac{S}{N}$ power ratio will always be greatest if the filter characteristic function, $Z(\nu)$, has the same shape as the frequency content of the signal to be received, $C(\nu)$."

This is the Matched Filter Theorem.

This has applications everywhere:

- radio receivers (tuners)
- astronomy

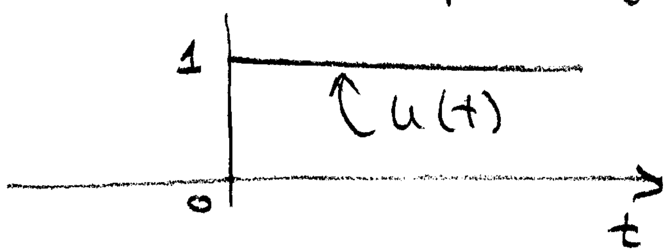
⋮

Nice theory but in practise you often don't know $C(\nu)$ since that is what you want to measure!

Be careful or your result becomes a self-fulfilling prophecy.

8.5 The Heaviside Step-function (James 4.7.1)

We need a signal to play with. We choose a simple off-on switch:



This is the Unit step we met in 2.2. It is often called the Heaviside Step-function.

Recall, $\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$

$$\Downarrow$$

$$\delta(t) = \frac{du(t)}{dt}$$

We need the F.T. of $u(t)$ so we can test ~~some~~ filters.

~~Define~~ $U(t) \rightleftharpoons \phi(\nu)$

ie $$U(t) = \int_{-\infty}^{\infty} \phi(\nu)\, e^{2\pi i \nu t} d\nu$$

Thus $$\frac{du(t)}{dt} \rightleftharpoons 2\pi i \nu\, \phi(\nu)$$

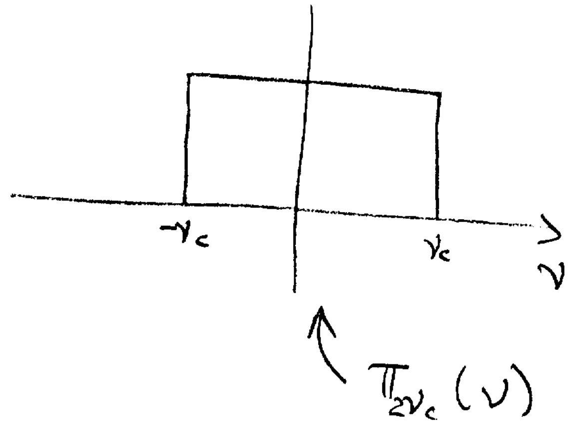
But $\frac{du(t)}{dt} = \delta(t)$ & we know $\delta(t) \rightleftharpoons 1$

$\therefore$ $1 = 2\pi i \nu\, \phi(\nu)$

which gives $\phi(\nu) = \frac{1}{2\pi i \nu}$.

Note that since $u(t)$ is ~~odd~~, $\phi(\nu)$ is imaginary.

8.6 Voltage Step Through a Simple Low-Pass Filter (James 4.7.2)



Let the input signal be $V_0 U(t)$ volts.

Thus

$$V_0 U(t) \rightleftharpoons \frac{V_0}{2\pi i\nu}$$

We apply the filter to frequency space:

$$V_{\text{filtered}}(\nu) = \frac{V_0}{2\pi i\nu}\,\Pi_{2\nu_c}(\nu)$$

To get back to time space:

$$v_{\text{filtered}}(t) = V_0\int_{-\nu_c}^{\nu_c}\frac{e^{2\pi i\nu t}}{2\pi i\nu}\,d\nu = V_0\int_{-\nu_c}^{\nu_c}\frac{\cos(2\pi\nu t)\,d\nu}{2\pi i\nu} + V_0 i\int_{-\nu_0}^{\nu_c}\frac{\sin(2\pi\nu t)\,d\nu}{2\pi i\nu}$$

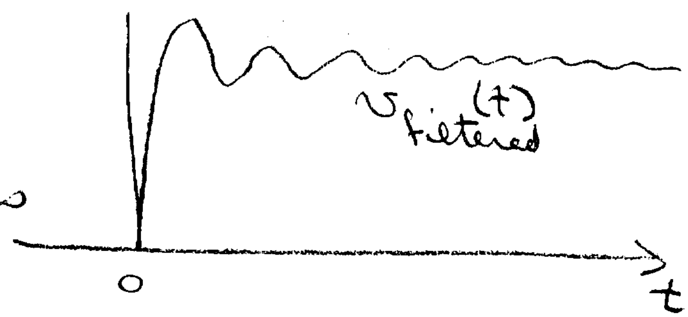
(note, $\Pi_{2\nu_c}$ has been replaced by finite limits).

Now, if you recall, the cosine integral $=0$ since $\frac{1}{2\pi i\nu}$ is odd.

$$\therefore\ v_{\text{filtered}}(t) = V_0\int_{-\nu_c}^{\nu_c}\frac{\sin(2\pi\nu t)\,d\nu}{2\pi\nu} = V_0 t\int_{-\nu_c}^{\nu_c}\operatorname{sinc}(2\pi\nu t)\,d\nu$$

$$= 2V_0 t\int_0^{\nu_c}\operatorname{sinc}(2\pi\nu t)\,d\nu$$

This must be numerically integrated to give $\rightarrow$

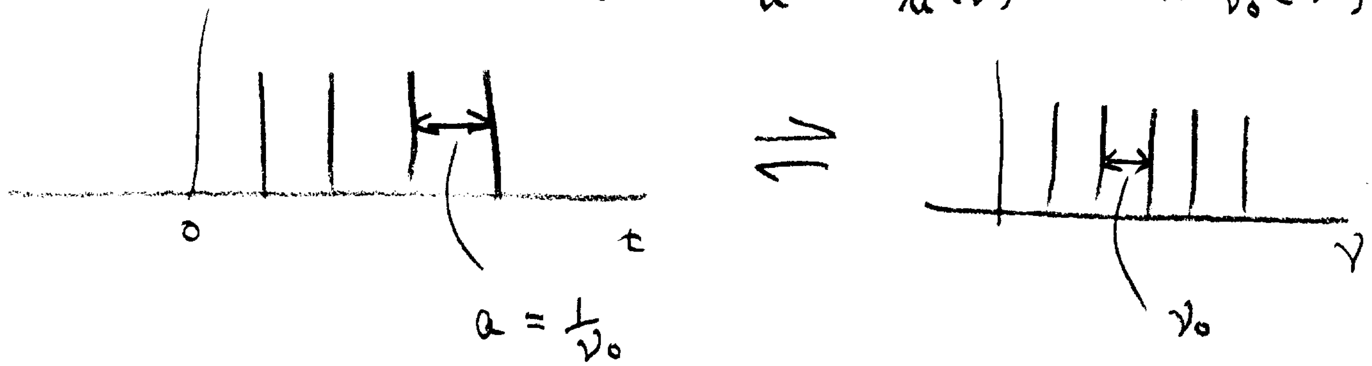


So cutting out the high frequencies clobbers the sharp edge!

8.7 Pulse Train Through a Simple Low-Pass filter (James 4.8.1)

The input voltage is now $III_a(t)$ and since

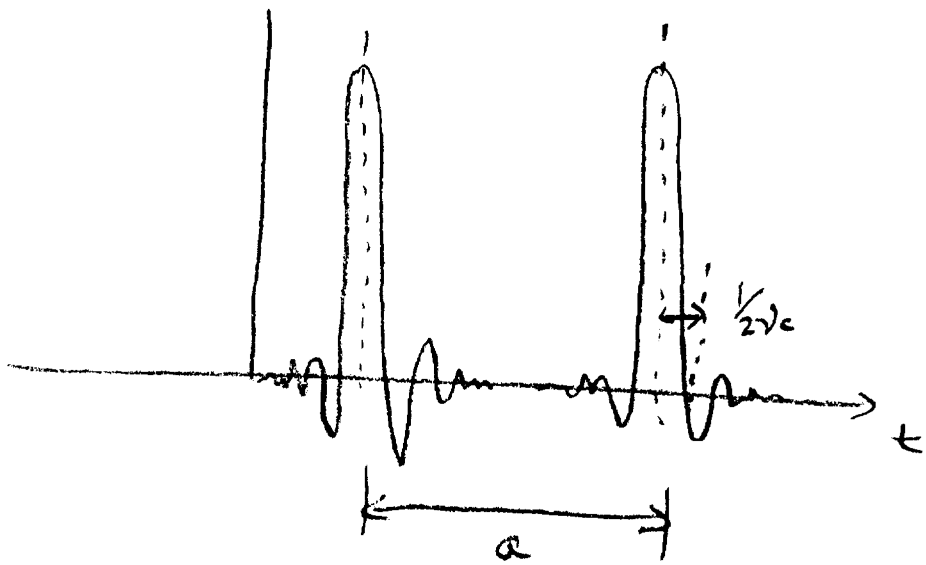
$$III_a(t) \rightleftharpoons \frac{1}{a} III_{1/a}(\nu) = \nu_0 III_{\nu_0}(\nu)$$



$$\therefore V_{filtered}(\nu) = \nu_0 III_{\nu_0}(\nu) \cdot \Pi_{2\nu_c}(\nu)$$

$$\Big\downarrow\!\Big\uparrow$$

$$\therefore v_{filtered}(t) = a\, III_a(t) * \mathrm{sinc}(2\pi\nu_c t)$$



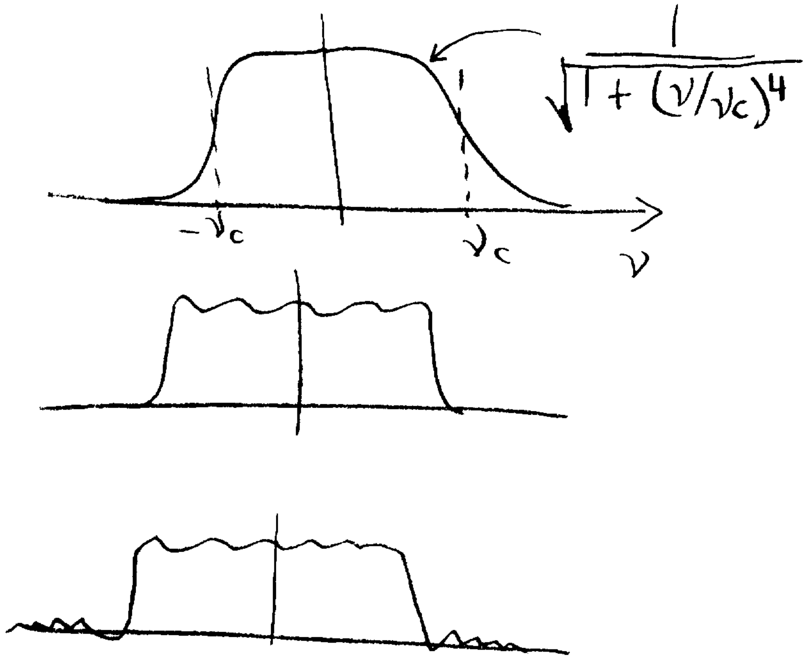
8.8 Practical Filters (Brach 3.3)

The ideal low-pass filter can be approximated by polynomials. Some common ones are:

Butterworth

Chebychev

Elliptic



Study of such filters is beyond the scope of this course.

8.9 Signal Modulation [See James 4.5 for a discussion.]

Consider a 'signal' $e^{2\pi i\nu_0 t}$. The F.T. of this is:

$$F(\nu) = \int_{-\infty}^{\infty} e^{2\pi i \nu_0 t} e^{-2\pi i \nu t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i(\nu-\nu_0)t}\, dt$$

$$= \delta(\nu-\nu_0)$$

ie $e^{2\pi i\nu_0 t} \rightleftharpoons \delta(\nu-\nu_0)$

Recall: we have seen:

$$\delta(t-t_0) \rightleftharpoons e^{-2\pi i\nu t_0}$$

Note the sign change when we 'flipped'. This is caused by the odd part of $e^{2\pi i\nu_0 t}$, ie the complex sine component.

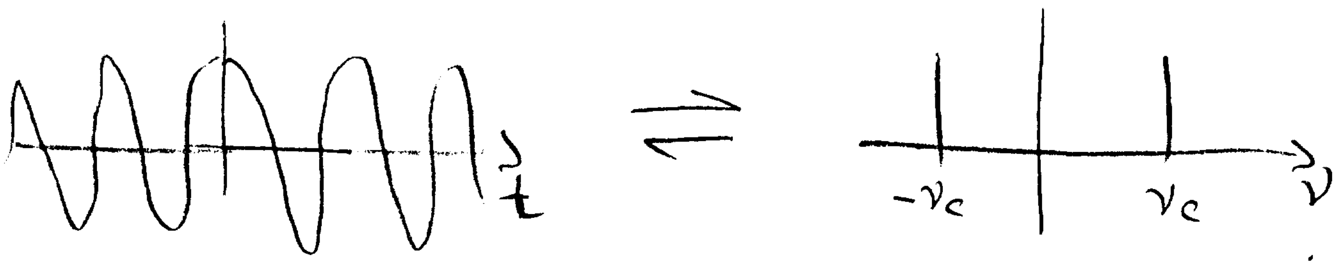
Now, since $\cos 2\pi\nu_0 t = \dfrac{e^{2\pi i\nu_0 t} + e^{-2\pi i\nu_0 t}}{2}$

we immediately have:

$$\cos 2\pi\nu_0 t \rightleftharpoons \frac{1}{2}\left[\delta(\nu-\nu_0) + \delta(\nu+\nu_0)\right]$$

[Of course, we could have deduced this immediately from $\delta(t-t_0) + \delta(t+t_0) \rightleftharpoons 2\cos(2\pi\nu t_0)$ that we derived in chapter 5]

So a constant signal (of amplitude 1) in the time domain modulated by a 'carrier wave', $\cos(2\pi\nu_c t)$ yields, in the ν domain



This is at the very heart of signal communication. We use a carrier signal to <u>shift the freq.</u> to anywhere we want – kHz range, MHz range, etc. So human voice (in the kHz range) can be sent on any number of 'channels'.

The 'signal' or information we wish to transmit is sent as an amplitude modulation (AM) or as a frequency modulation (FM) of the carrier cosine, ie:

$$V(t) = A \cos 2\pi\nu t$$

vary this for FM
$\nu = f_n(t)$

vary this for AM
$A = f_n(t)$

The <u>phase</u> of a signal is defined as

$$\underline{\Phi} = 2\pi \nu_0 t + \phi(t)$$

$\uparrow$ phase $\qquad\qquad$ $\uparrow$ phase angle

The instaneous frequency, ν is

$$\nu = \frac{1}{2\pi}\frac{d\Phi}{dt} = \nu_0 + \frac{1}{2\pi}\frac{d\phi}{dt}$$

So we think of a varying ν as a varying ϕ. Hence frequency modulation is often called phase modulation. The two are the same.

See Fante, section 1.7 for more on this.

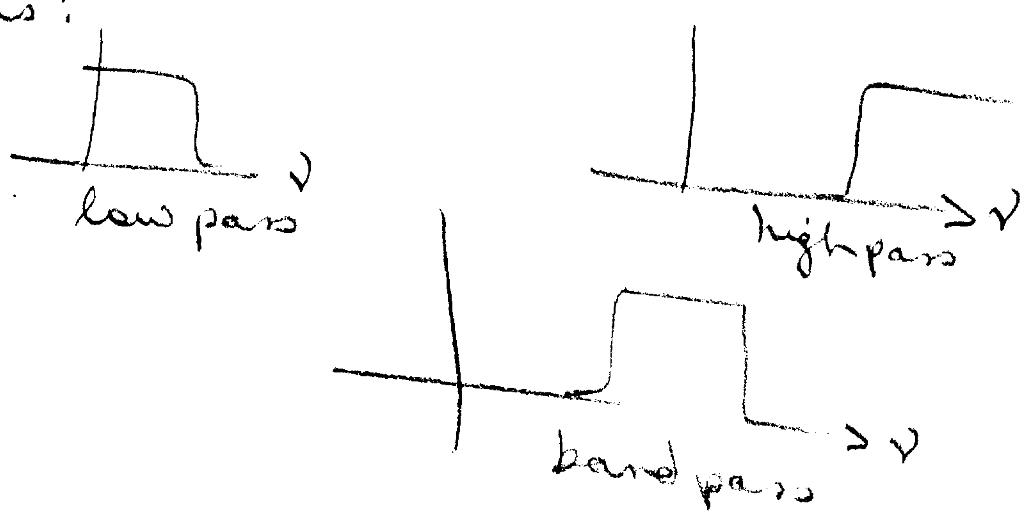
8.10 Recap

$$\int_{-\infty}^{\infty} V^2(t)\,dt = \int_{-\infty}^{\infty} \underbrace{|c(\nu)|^2}_{SPD}\,d\nu$$

$$A(\tau) \equiv \int_{-\infty}^{\infty} v(t)\,v(t+\tau)\,d\tau = \text{autocorrelation}$$

SPD is the FT of the autocorrelation

Filters:

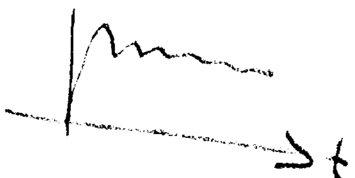


Use filters to clean up noise.

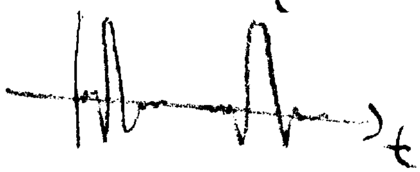
Best S/N when $\underline{z(\nu)} = c(\nu)$ – Matched filter

$u(t) \rightleftharpoons \dfrac{1}{2\pi i\nu}$ Heaviside function (unit step)

Low pass + step:

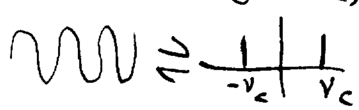


Low pass + pulses:



Signal Modulation

$\cos 2\pi\nu_c t$
$\rightleftharpoons \frac{1}{2}[\delta(\nu-\nu_c) + \delta(\nu+\nu_c)]$



Practical filters: Butterworth, Chebyshev ...