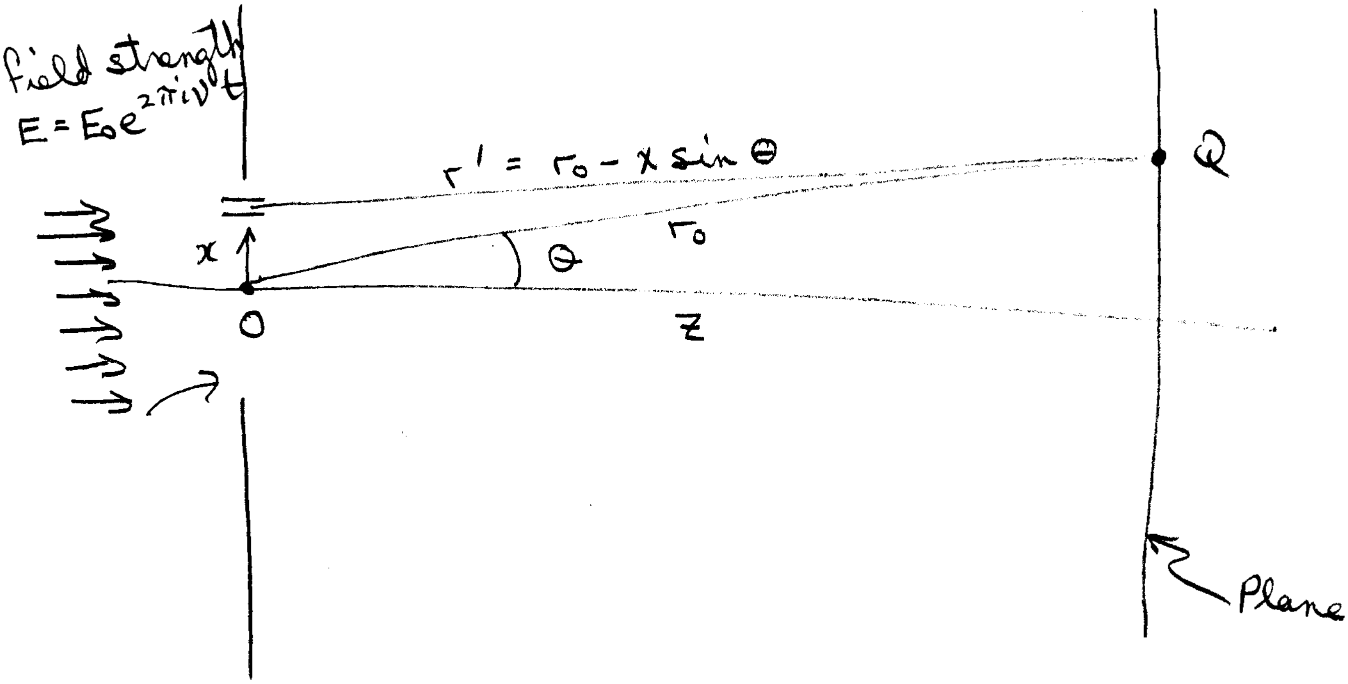
Chapter 7 ~~Application~~: Fraunhofer Diffraction

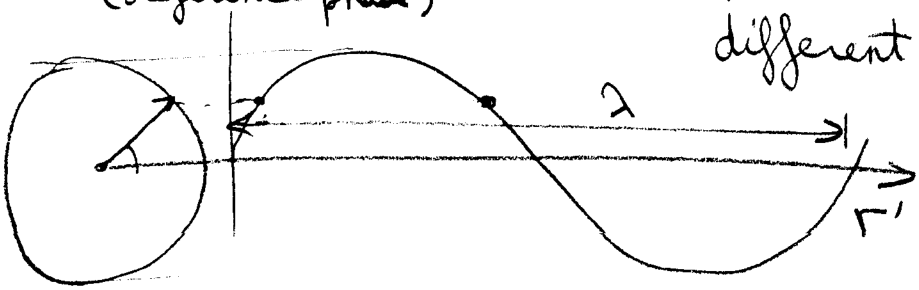
(following James, chapter 3)

Consider a coherent light source shining through an aperture:



The field intensity at a point, Q, on the plane due to the light coming from a segment of the aperture, dx is:

$$dE(Q) = E_0 \underbrace{e^{2\pi i\nu t}}_{\text{set this to 1 at } t=0 \text{ (reference phase)}} \underbrace{e^{+2\pi i r'/\lambda}}_{\text{phase shift due to travelling different distances.}}$$



Integrating over the aperture:

$$E(Q) = \int_{\text{aperture}} E_0 \, e^{2\pi i (r_0 - x\sin\theta)/\lambda} \, dx$$

$$= E_0 \, e^{2\pi i r_0/\lambda} \int_{\text{aperture}} e^{-x\sin\theta/\lambda} \, dx$$

At a given angle, θ, $\sin\theta/\lambda$ = constant $\equiv p$.

$$\therefore \; E(Q) \equiv E = E_0 \, e^{2\pi i r_0/\lambda} \int_{-\infty}^{\infty} A(x) e^{-2\pi i p x} \, dx$$

where $A(x) \equiv$ aperture function (in this case, $A(x) = \Pi_a(x)$, ie a slit of width a).

So the field at the plane is the <u>Fourier Transform</u> of the source field at the aperture.

Let's look at a few types of slits.

Single slit

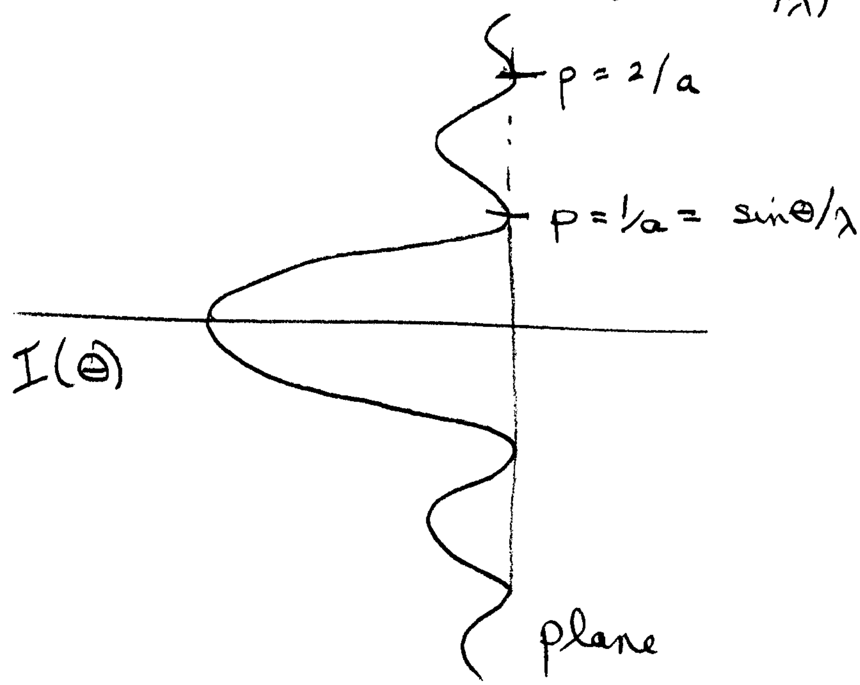
$$A(x) = \Pi_a(x)$$

$$\therefore\ E = E_0 e^{2\pi i r_0/\lambda} \int_{-\infty}^{\infty} \Pi_a(x)\, e^{-2\pi i p x}\, dx$$

$$= \underbrace{E_0 e^{2\pi i r_0/\lambda}\, a}_{k}\ \mathrm{sinc}(\pi p a)$$

$$= k\ \mathrm{sinc}(\pi p a) = k\ \mathrm{sinc}(\pi a \sin\theta/\lambda)$$

The intensity is $E E^* = I(\theta) = |k|^2\, \mathrm{sinc}^2(\pi a \sin\theta/\lambda)$



2 Point Sources at $\pm b/2$

(example: 2 antennae transmitting in phase)

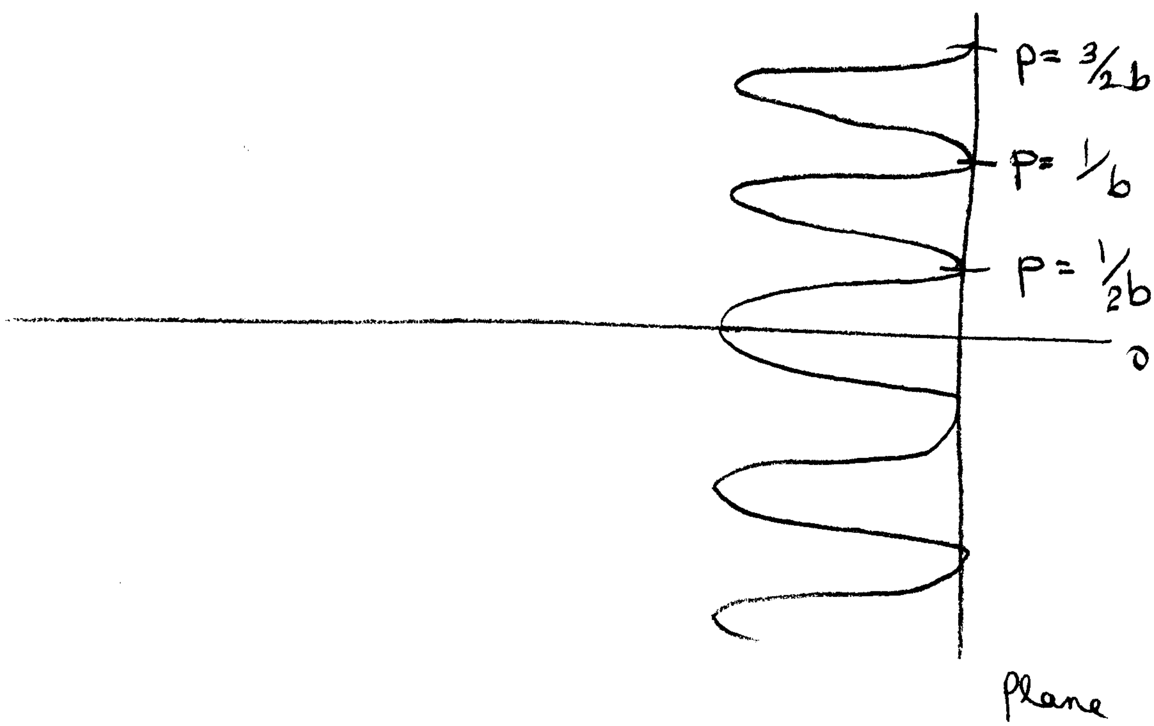
$$A(x) = \delta(x - b/2) + \delta(x + b/2)$$

$$\& \quad E = E_0 \, e^{2\pi i r_0/\lambda} \int_{-\infty}^{\infty} A(x)\, e^{-2\pi i p x}\, dx$$

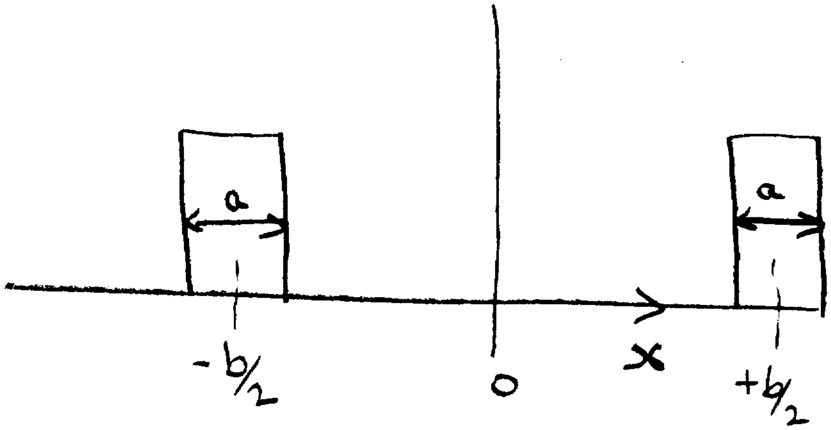
Recall: $A(x) \rightleftharpoons 2\cos(\pi b p)$

$$\therefore \quad E = 2k \cos(\pi b p) = 2k\cos(\pi b \sin\theta/\lambda)$$

$$\& \quad I(\theta) = 4|k|^2 \cos^2(\pi b \sin\theta/\lambda)$$



2 slits, each of width a, centres at $\pm b/2$



$$A(x) = \Pi_a(x - b/2) + \Pi_a(x + b/2)$$
$$= \int_{-\infty}^{\infty} \Pi_a(x') \left[\delta(x - (x' - b/2)) + \delta(x + (x' - b/2)) \right] dx'$$

(Recall: $\int_{-\infty}^{\infty} f(x')\,\delta(x - x')\,dx' = f(x)$)

But these integrals are just convolutions.

(Recall: $f * g = \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau$)

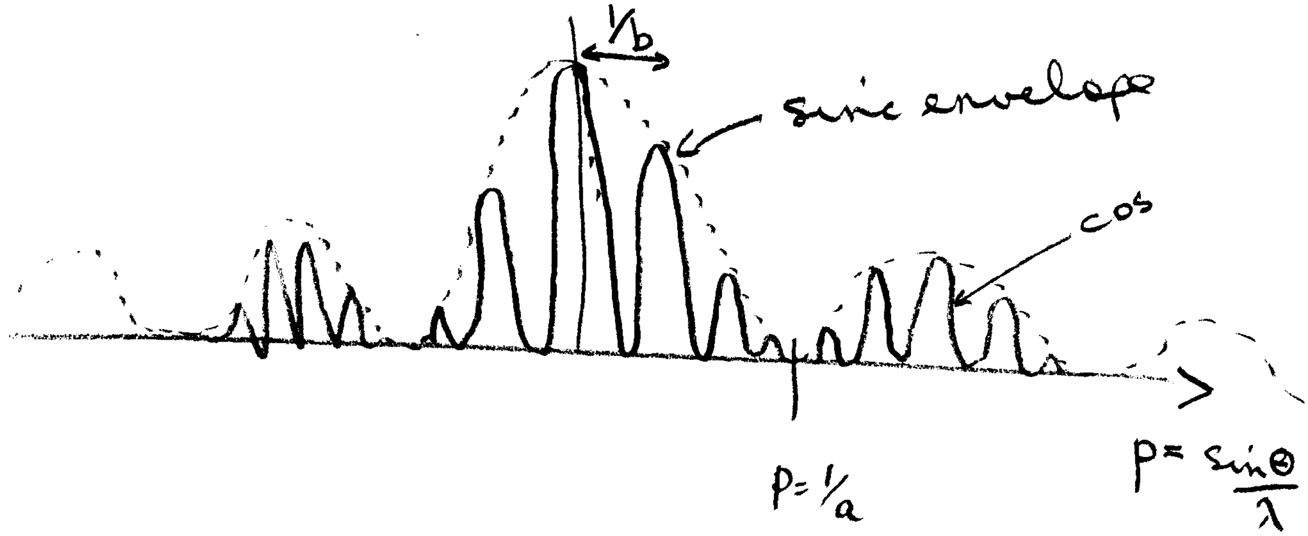
$$\therefore\ A(x) = \Pi_a(x) * \left[\delta(x - b/2) + \delta(x + b/2) \right]$$

& since $f * g \rightleftharpoons F \cdot G$, we have

$$E = 2k\ \underbrace{\mathrm{sinc}(\pi a \sin\theta/\lambda)}_{\text{from the F.T. of } \Pi_a(x)}\ \underbrace{\cos(\pi b \sin\theta/\lambda)}_{\text{from the F.T. of the } \delta\text{'s.}}$$

& $I(\theta) = E E^*$ as usual

For the 2 slits the interference pattern is

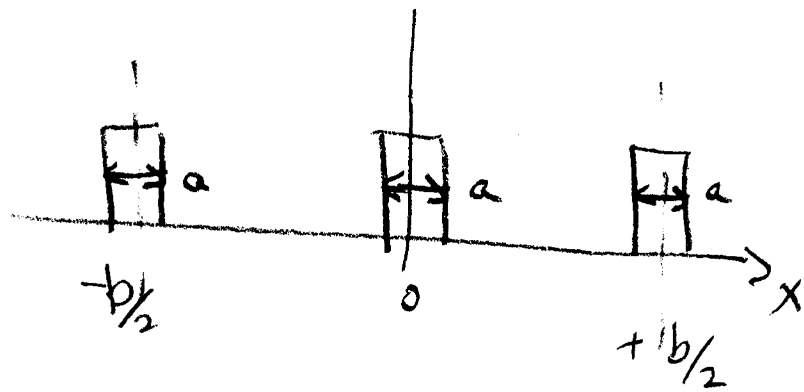


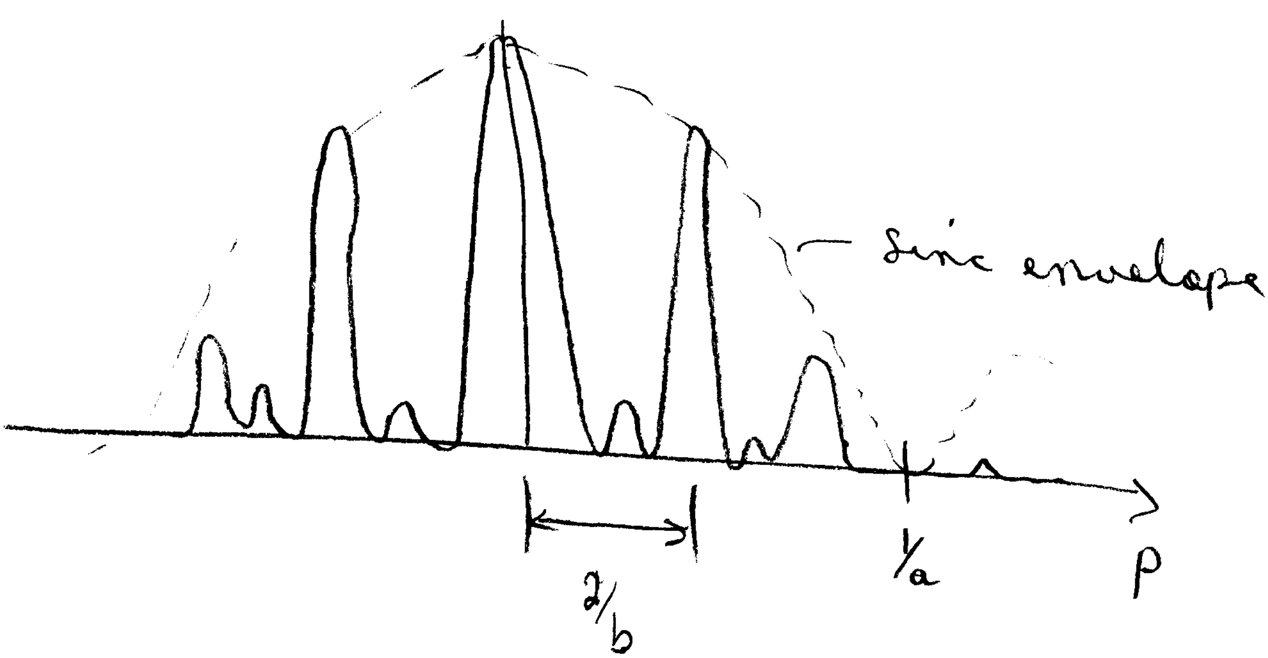
3 slits (as above but with extra slit at $x=0$)
(note : James uses a different separation, b)

$$A(x) = \Pi_a(x) * \left[\delta(x - b/2) + \delta(x) + \delta(x + b/2) \right]$$

$$\therefore E = k \,\text{sinc}(\pi p a) \left[e^{-\pi i b p} + 1 + e^{\pi i p b} \right]$$

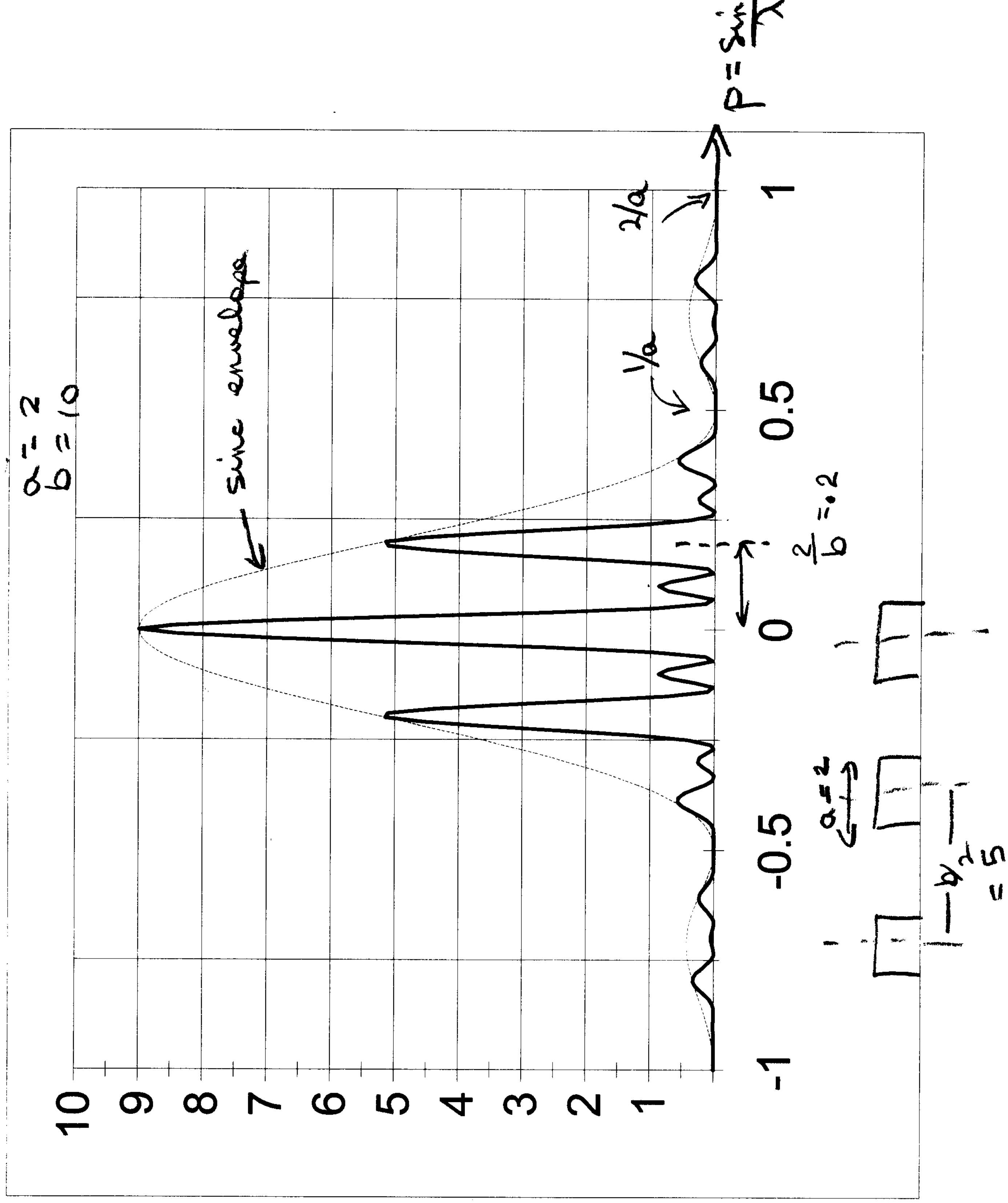
$$= k \,\text{sinc}(\pi p a) \left[2\cos(\pi p b) + 1 \right]$$





(see spreadsheet output on next page)

3 slit example



Recap

The diffraction pattern from a coherent source is the Fourier Transform of the aperture:

$$E = E_0 e^{2\pi i r_0/\lambda} \int_{-\infty}^{\infty} A(x) e^{-2\pi i p x} \, dx$$

where $p = \dfrac{\sin\theta}{\lambda}$

The geometry of the aperture determines the pattern.