Mathematics - Course 221

DIFFERENTIATING EXPONENTIAL FUNCTIONS

I Derivative of $e^{g(x)}$

Recall (lesson 221.20-2) that the derivative of the function f(x) is the 'instantaneous' rate of change of f(x) with respect to x.

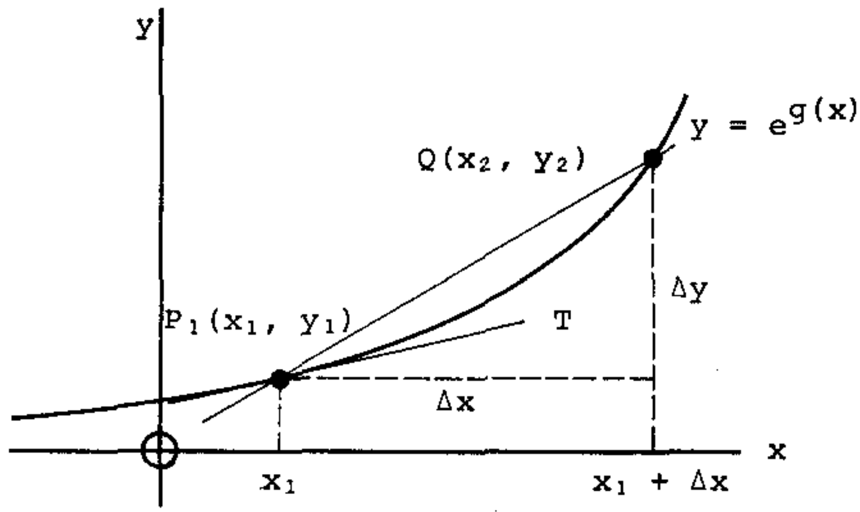


Figure 1

In Figure 1, the 'instantaneous' R/C $f(x) = e^{g(x)}$ wrt x at $x = x_1$ is equivalent to

(1) $\lim_{Q \to P_1}$ (slope of secant P_1Q)

(2) slope of tangent P_1T

(3) $f^1(x_1)$, the derivative of $e^{g(x)}$ evaluated at $x = x_1$.

Recall (lesson 221.20-2) the basic *defining equation of the derivative of f(x):*

$$f^1(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Applying this equation to $f(x) = e^{g(x)}$ yields

$$f^1(x) = \lim_{\Delta x \to o} \frac{e^{g(x + \Delta x)} - e^{g(x)}}{\Delta x}$$

It can be shown (but is beyond the scope of this course to do so) that the above limit reduces to

$$e^{g(x)} \; g^1(x)$$

Hence the formula for the *derivative of an exponential function* is

$$\frac{d}{dx} e^{g(x)} = e^{g(x)} \; g^1(x)$$

Example 1

$$\frac{d}{dx} e^x = e^x \frac{d}{dx} x$$

$$= e^x$$

Note that e^x equals its own derivative!

Example 2

$$\frac{d}{dx} 6e^{x^2} = 6 \frac{d}{dx} e^{x^2}$$

$$= 6e^{x^2} \frac{d}{dx} x^2$$

$$= 6e^{x^2} (2x)$$

$$= 12xe^{x^2}$$

Example 3

$$\frac{d}{dx} e^{2\sqrt{x}} = e^{2\sqrt{x}} \frac{d}{dx} 2\sqrt{x}$$

$$= e^{2\sqrt{x}} \left(2\frac{d}{dx} x^{1/2}\right)$$

$$= e^{2\sqrt{x}} (\not{2}) \left(\frac{1}{\not{2}} x^{-1/2}\right)$$

$$= \frac{e^{2\sqrt{x}}}{\sqrt{x}}$$

Example 4

$$\frac{d}{dx} (15x^3 - e^{-ax^2}) = \frac{d}{dx} 15x^3 - \frac{d}{dx} e^{-ax^2}$$

$$= 15 \frac{d}{dx} x^3 - e^{-ax^2} \frac{d}{dx} (-ax^2)$$

$$= 15(3x^2) - e^{-ax^2} \left(-a \frac{d}{dx} x^2\right)$$

$$= 45x^2 + 2axe^{-ax^2}$$

Example 5

Given the displacement function

$$s(t) = 5t^2 + 100e^{-0.4t},$$

(a) find the velocity function $v(t)$

(b) find the acceleration function $a(t)$

(c) sketch the graphs of $s(t)$, $v(t)$ and $a(t)$ over the interval $0 \le t \le 10$

Solution

(a) $v(t) = s^1(t)$

$$= \frac{d}{dt}(5t^2 + 100e^{-0.4t})$$

$$= 10t + 100e^{-0.4t}\frac{d}{dt}(-0.4t)$$

$$= \underline{\underline{10t - 40e^{-0.4t}}}$$

(b) $a(t) = \frac{dv}{dt}$

$$= 10\frac{d}{dt}t - 40\frac{d}{dt}e^{-0.4t}$$

$$= 10 - 40e^{-0.4t}\frac{d}{dt}(-0.4t)$$

$$= \underline{\underline{10 + 16e^{-0.4t}}}$$

t	0	1	2	3	4	6	8	10
s	100	72	65	75	100	189	324	502
v	-40	-17	2	18	32	56	78	99
a	26	20.7	17.2	14.8	13.2	11.5	10.7	10.3

The following are sample calculations of those used to produce the above table of values:

$$s(10) = 5(10)^2 + 100e^{-0.4(10)}$$

$$= 500 + 100e^{-4}$$

$$= 500 + 100\ (0.018)$$

$$= \underline{\underline{501.8}}$$

$$v(10) = 10(10) - 40e^{-0.4(10)}$$

$$= 100 - 40\ (0.018)$$

$$= \underline{\underline{99.3}}$$

$$a(10) = 10 + 16e^{-0.4(10)}$$

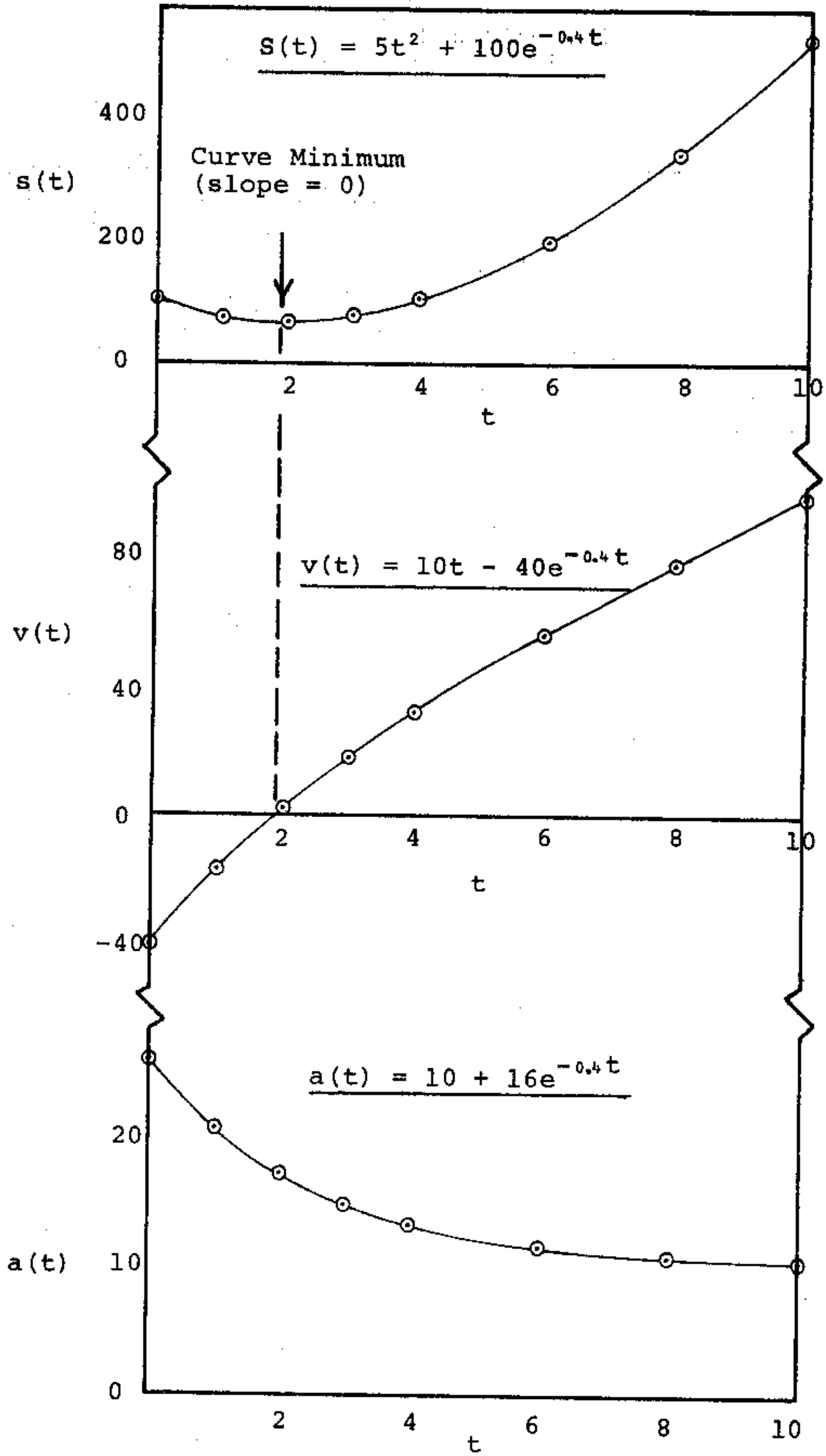
$$= 10 + 16\ (0.018)$$

$$= \underline{\underline{10.3}}$$

It was stated in lesson 221.20-3 that velocity is the slope of the s-t curve, and that acceleration is the slope of the v-t curve. Are these statements consistent with the curves of Figure 2?

Note that the slope of the s-t curve is negative at t = 0, rises to zero at the curve minimum (t = 1.9), and then increases positively to t = 10. Note that this is precisely the behaviour of the v-t curve.

Note that the v-t curve rises most sharply at t = 0, and gradually settles to a slower, almost linear rate of rise. Accordingly one would expect a positive acceleration in the entire interval $0 \le t \le 10$, and one that would fall from its initial value towards a constant value. This is precisely the behaviour of the a-t curve.



$S(t) = 5t^2 + 100e^{-0.4t}$
400
s(t)
Curve Minimum
(slope = 0)
200
0
2
4
6
8
10
t
80
$v(t) = 10t - 40e^{-0.4t}$
v(t)
40
0
2
4
6
8
10
t
-40
$a(t) = 10 + 16e^{-0.4t}$
20
a(t)
10
0
2
4
6
8
10
t

Figure 2

II Application to Nuclear Decay

The number of radioactive atoms remaining in a radioactive source decays exponentially with time, according to the relation.

$$N(t) = N_0 e^{-\lambda t}$$

where $N(t)$ = number of radioactive atoms remaining after t seconds,

N_0 = number of radioactive atoms at time $t = 0$, and

λ is the decay constant of the radionuclide in s^{-1}

To find the R/C N wrt t, ie, the number of nuclei decaying per second, differentiate the above relation wrt time:

$$\frac{dN}{dt} = \frac{d}{dt} N_0 e^{-\lambda t}$$

$$= N_0 \frac{d}{dt} e^{-\lambda t} \qquad (N_0 \text{ a constant})$$

$$= N_0 \, e^{-\lambda t} \frac{d}{dt} (-\lambda t)$$

$$= N_0 \, e^{-\lambda t} (-\lambda \frac{d}{dt} t)$$

$$= -\lambda N_0 e^{-\lambda t}$$

$$\therefore \quad \boxed{\frac{dN}{dt} = -\lambda N}$$

Note that $\frac{dN}{dt}$ stands for the rate of increase in N. Hence $\frac{dN}{dt}$ is negative (see minus sign on RHS), since N is actually decreasing.

The number of nuclei decaying per unit time is called the *activity* of a source.

Example 6

How many radioactive nuclei are required to make a 5 mCi source of a nuclide whose decay constant equals $7.3 \times 10^{-5}\ s^{-1}$? (1 curie = 3.7×10^{10} dps)

Solution

$$\frac{dN}{dt} = -5 \text{ mCi}$$

$$\Rightarrow \quad -\lambda N = -5 \times 10^{-3} \times 3.7 \times 10^{10}$$

$$\therefore \quad N = \frac{5 \times 10^{-3} \times 3.7 \times 10^{10}}{7.3 \times 10^{-5}}$$

$$= \underline{\underline{2.5 \times 10^{12}}}$$

ie, there are 2.5×10^{12} atoms in a 5 mCi Source.

* * *

If source activity is designated "A",

then $A(t) = -\frac{dN}{dt}$ (rate of decrease in N)

$$= \lambda N$$

$$= \lambda N_0 e^{-\lambda t} \qquad (\therefore \quad N = N_0 e^{-\lambda t})$$

then $A(0) = \lambda N_0 e^0$

$$= \lambda N_0$$

Let $A_0 = A(0)$

Then $A_0 = \lambda N_0$

and $$\boxed{A(t) = A_0 e^{-\lambda t}}$$

ie, the activity A(t) obeys the same exponential relationship as N(t).

Example 7

Find the time required for the activity of a source of decay constant 3.5×10^{-4} s^{-1} to decay by a factor of 1000.

Solution

Let required time be t_1.

Then $A(t_1) = A_0 e^{-\lambda t_1}$

ie, $\frac{A(t_1)}{A_0} = e^{-\lambda t_1}$

$\therefore \quad e^{-\lambda t_1} = 0.001 \qquad (\because \frac{A(t_1)}{A_0} = \frac{1}{1000})$

Taking natural log of both sides,

$$\ell n\, e^{-\lambda t_1} = \ell n\, 0.001$$

$\therefore \quad -\lambda t_1 = \ell n\, 10^{-3}$ (cf lesson 321.10-4)

$$\therefore \quad t_1 = \frac{\ell n\, 10^{-3}}{-\lambda}$$

$$= \frac{-6.91}{-3.5 \times 10^{-4}}$$

$$= \underline{\underline{2.0 \times 10^4 \text{ seconds or } 5.5 \text{ hours}}}$$

Example 8

Prove: $t_{1/2} = \frac{0.693}{\lambda}$, where $t_{1/2}$ is the half-life of a radionuclide, ie, the time required for source activity to decay to one-half its original activity.

Solution

$$A(t_{1/2}) = A_0 e^{-\lambda t_{1/2}}$$

$$\therefore \quad \frac{A(t_{1/2})}{A_0} = e^{-\lambda t_{1/2}}$$

$$\therefore \quad e^{-\lambda t_{1/2}} = 0.5 \qquad \left(\because \frac{A(t_{1/2})}{A_0} = 0.5\right)$$

$$\therefore \quad \ln e^{-\lambda t_{1/2}} = \ln 0.5$$

$$\therefore \quad -\lambda t_{1/2} = -0.693$$

$$\therefore \quad t_{1/2} = \frac{0.693}{\lambda}$$

III Application to Reactor Power Growth

Reactor power grows exponentially in time, approximately according to the relation,

$$P(t) = P_0 e^{\frac{\Delta k}{L} t}$$

where $P(t)$ is reactor power at time t,

P_0 is reactor power at $t = 0$,

Δk is the reactivity in units of "k",

L is the mean neutron lifetime in the reactor.

For example, if $P_0 = 100$ W, and $\frac{\Delta k}{L} = 0.05$, the graph of $P(t)$ vs t is shown in Figure 3.

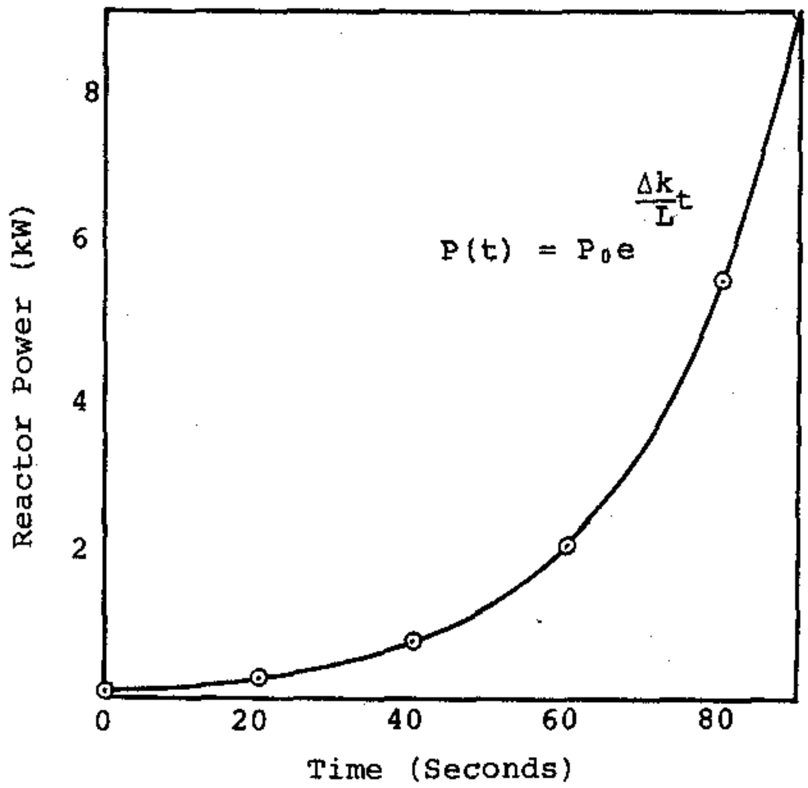


Figure 3

DEFINITION: The *reactor period* "T" is the time required for the power to increase by a factor of e.

Proof that Reactor Period $= \frac{L}{\Delta k}$

$$P(T) = eP_0 \quad \text{by definition of T}$$

ie, $$e\not{P}_0 = \not{P}_0 e^{\frac{\Delta k}{L} T}$$

$\therefore$ $$\ln e = \ln e^{\frac{\Delta k}{L} T}$$

ie, $$1 = \frac{\Delta k}{L} T$$

$\therefore$ $$\boxed{T = \frac{L}{\Delta k}}$$

$\therefore$ an alternative form of the power growth equation is

$$\boxed{P(t) = P_0 e^{t/T}}$$

from which it is obvious that each time t increases by T, P increases by a factor of e, consistent with previous definition of T.

Not only does the power P(t) grow exponentially with time, but so also does the rate of growth, $P^1(t)$, as shown below:

$$\frac{dP}{dt} = \frac{d}{dt} P_0 e^{\frac{\Delta k}{L} t}$$

$$= P_0 \frac{d}{dt} e^{\frac{\Delta k}{L} t}$$

$$= P_0 e^{\frac{\Delta k}{L} t} \frac{d}{dt} \frac{\Delta k}{L} t$$

$$= P_0 e^{\frac{\Delta k}{L} t} \frac{\Delta k}{L} \frac{d}{dt} t$$

$$= \frac{\Delta k}{L} \underbrace{P_0 e^{\frac{\Delta k}{L} t}}_{P(t)}$$

$$\therefore \quad \boxed{\frac{dP}{dt} = \frac{\Delta k}{L} P(t) = \frac{1}{T} P(t)}$$

Note that power growth rate $P^1(t)$ is directly proportional to product of reactivity Δk and power P(t). Therefore, given sufficiently high values of Δk and P, P^1 may be so high that rated power is exceeded before the regulation system can arrest power growth.

Thus, for reactor protection, a signal is required to detect dangerously high reactivity valves at low power. Such a signal is one whose output varies as the rate of change of the logarithm of reactor power. This signal is known as "*rate log power*":

$$\frac{d}{dt}(\ell n\ P(t)) = \frac{d}{dt}\ \ell n\ P_0 e^{\frac{\Delta k}{L}t}$$

$$= \frac{d}{dt}(\ell n\ P_0 + \ell n\ e^{\frac{\Delta k}{L}t})$$

$$= \frac{d}{dt}\ \ell n\ P_0 + \frac{d}{dt}\frac{\Delta k}{L}t$$

$$= 0 + \frac{\Delta k}{L}\frac{d}{dt}t$$

$$= \frac{\Delta k}{L}$$

$$\therefore \quad \boxed{\text{rate log power, } \frac{d}{dt}(\ell n\ P(t)) = \frac{\Delta k}{L} = \frac{1}{T}}$$

Note that rate log power is proportional to reactivity Δk, independent of reactor power. Hence the reactor can be tripped by this signal at low power, eg, 0.001% full power, long before the *linear rate power*, $P^1(t)$ gets out of hand.

ASSIGNMENT

1. Differentiate:

(a) e^{x^2-4} (b) $-e^{-x}$

(c) $-e^{-x^{-1}}$ (d) $2e^{-1/\sqrt{x}}$

(e) $5e^{\sqrt{t}}\ (t^2-1)$ (f) $\frac{1}{3}\ e^{-1/x^3}$

2. Find (i) v(t) (ii) a(t) (iii) v(2) if

(a) $s(t) = e^t - t^3$

(b) $s(t) = e^{-t} + 2t$

3. Plot s - t, v - t, a - t curves for the displacement function of 2(a) above over the time interval $0 \leq t \leq 3$. Do the slopes of the s - t and v - t curves appear to verify the definitions, $v(t) = s^1(t)$ and $a(t) = v^1(t)$, respectively?

4. If 2.0×10^{19} radioactive nuclei constitute a 5.0 mCi source, what is the decay constant of the radionuclide? (1 curie = 3.7×10^{10} dps)

5. (a) What is the activity of a source consisting of 7.0×10^{13} radioactive nuclei, and having decay constant 2.4×10^{-4} s^{-1}?

(b) How many radioactive nuclei remain after (i) 20 minutes? (ii) 6 half-lives?

(c) Calculate the source activity after (i) 20 minutes (ii) 6 half-lives.

(d) Calculate the half-life of the source.

(e) How long does the source take to decay to 10 mCi?

6. If $N(t) = N_0 e^{-\lambda t}$ and $A = -\frac{dN}{dt}$, prove that (a) $A = \lambda N$

(b) $A(t) = A_0 e^{-\lambda t}$

7. Prove that $t_{1/2} = \frac{\ln 2}{\lambda}$.

8. If $P(t) = P_0 e^{t/T}$, prove that

(a) $P^1(t) = \frac{1}{T} P(t)$

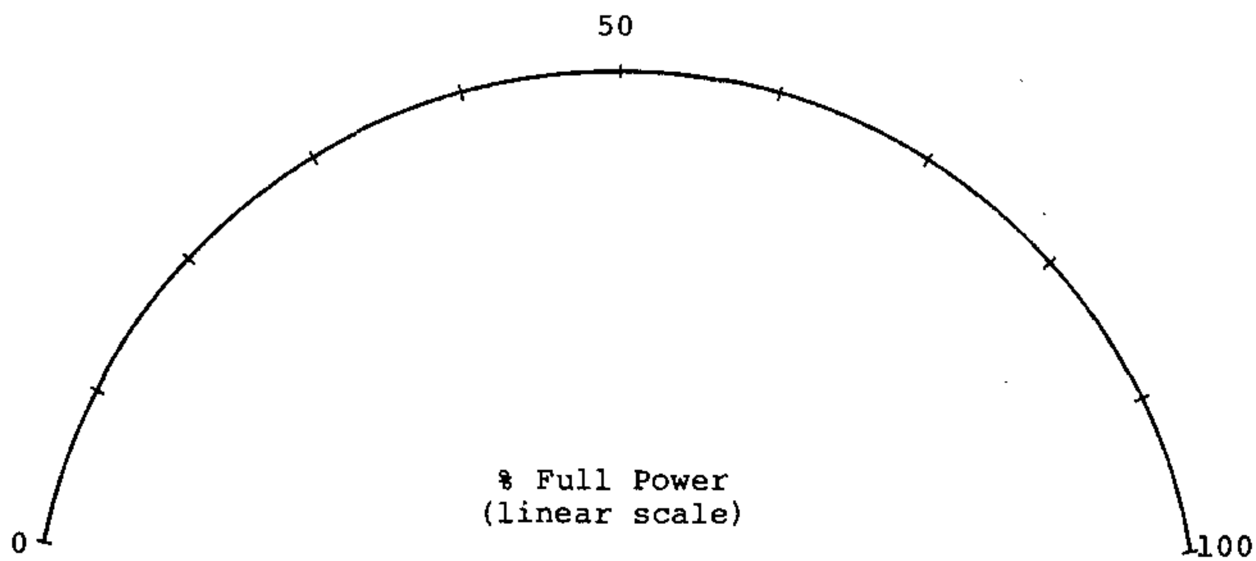
(b) $\frac{d}{dt} \ln P(t) = \frac{1}{T}$

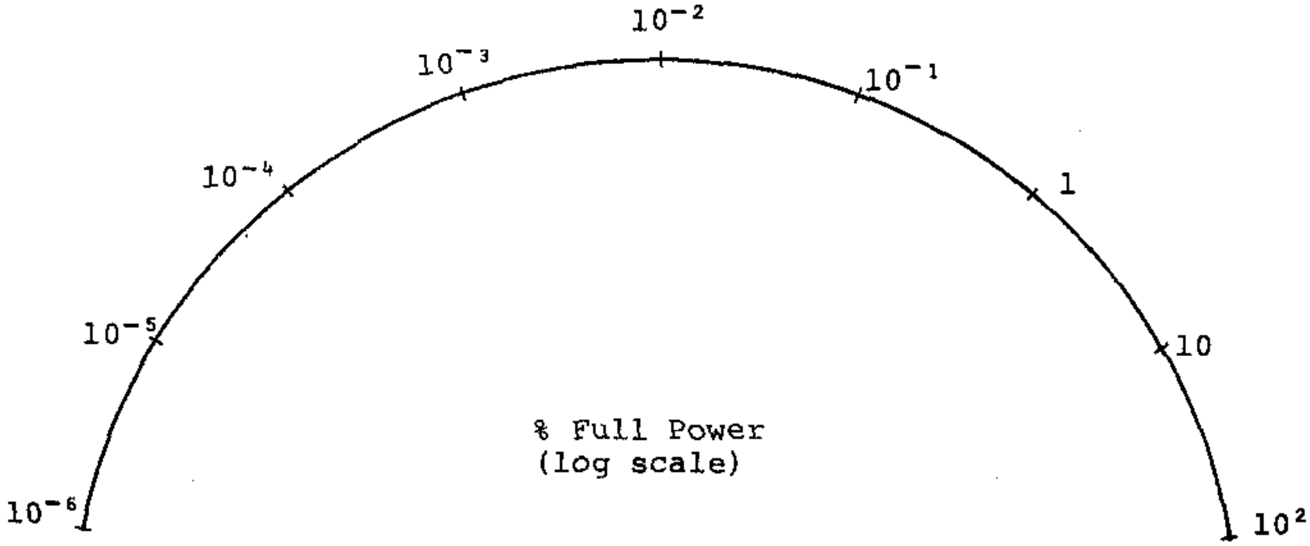
9. Plot a graph of N(t) vs t over the interval $0 \leq t \leq 18$ hours if $N(t) = N_0 e^{-\lambda t}$, where $N_0 = 10^{20}$ and $\lambda = 6.4 \times 10^{-5}\ s^{-1}$.

(a) on linear paper (b) on log-linear paper.

10. (a) Make a table of values of reactor power P(t) and linear rate, $P^1(t)$ with 20-second increments in t over the interval $0 \leq t \leq 5$ minutes. Assume $P_0 = 100$ W and $\frac{\Delta k}{L} = 0.05$. Express P and P^1 in units of % full power, assuming full power equals 100 MW.

(b) Show consecutive positions of indicating needles on the following meters, at 20-second intervals.





(c) Describe the needle's motion across each of the above scales, and relate descriptions to the mathematical expressions for linear rate and rate log power.

(d) Which meter is more suitable for monitoring power at low power levels? At high power levels?

(e) Which of the following signals is more appropriate for reactor power control

(i) at low power levels?

(ii) at high power levels?

a signal whose output is proportional to reactor power P, or one whose output is proportional to the logarithm of reactor power, log P?

11. Explain the advantage of a rate log signal for reactor protection.

12. Show that $\frac{d}{dt}$ (log P(t)) = $\frac{\Delta k}{L}$ log e, where log P is the <u>common</u> logarithm of P.

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