Mathematics - Course 221

THE DERIVATIVE

I LINEAR FUNCTIONS

Recall that *linear functions* are functions of the form

$$f(x) = mx + b,$$

where "m" is the slope, and "b" is y-intercept of the line $y = f(x)$.

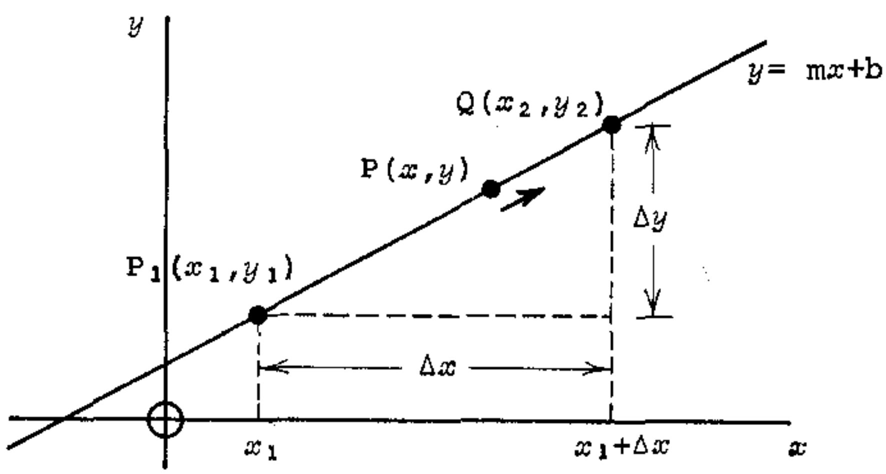

Figure 1

For example, as the point $P(x,y)$ moves up the line from P_1 to Q in Figure 1, x increases by Δx and y increases by Δy, and y increases m times as fast as x, where

$$m = \frac{\Delta y}{\Delta x}$$

ie, for a line with slope 2, y increases twice as fast as x as point $P(x,y)$ moves along the line.

In other words, the slope of a line gives the rate of change of y with respect to x along the line.

In Figure 1, as P moves from P_1 to Q, x and y are both continually changing. Therefore the rate of change of y with respect to x (the slope) must have meaning not only over the whole segment from P_1 to Q, but at every point along the line. The slope of the line at a specific point P_1 may be called the *'instantaneous' rate of change of y with respect to x at P_1.*

Note that "instantaneous" is placed in inverted commas since $x = x_1$ represents an instant only in a figurative sense.

The slope of the line at point P_1 is found by *taking the limit* of the slope of segment P_1Q as Q moves to P_1 along the line,

ie, symbolically,

$$\text{slope of line at } P_1 = \lim_{Q \to P_1} \text{slope segment } P_1Q$$

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Note: Read "$\lim_{Q \to P_1}$" as "limit as Q tends to P_1 of..."

and "$\lim_{\Delta x \to 0}$" as "limit as Δx tends to zero of..."

Example 1

Find the 'instantaneous' rate of change of $f(x) = 2x + 1$ with respect to x at $x = 3$.

Solution

The problem may be restated as follows: "Find the slope of the line $y = 2x + 1$ at the point $P_1(3,7)$".

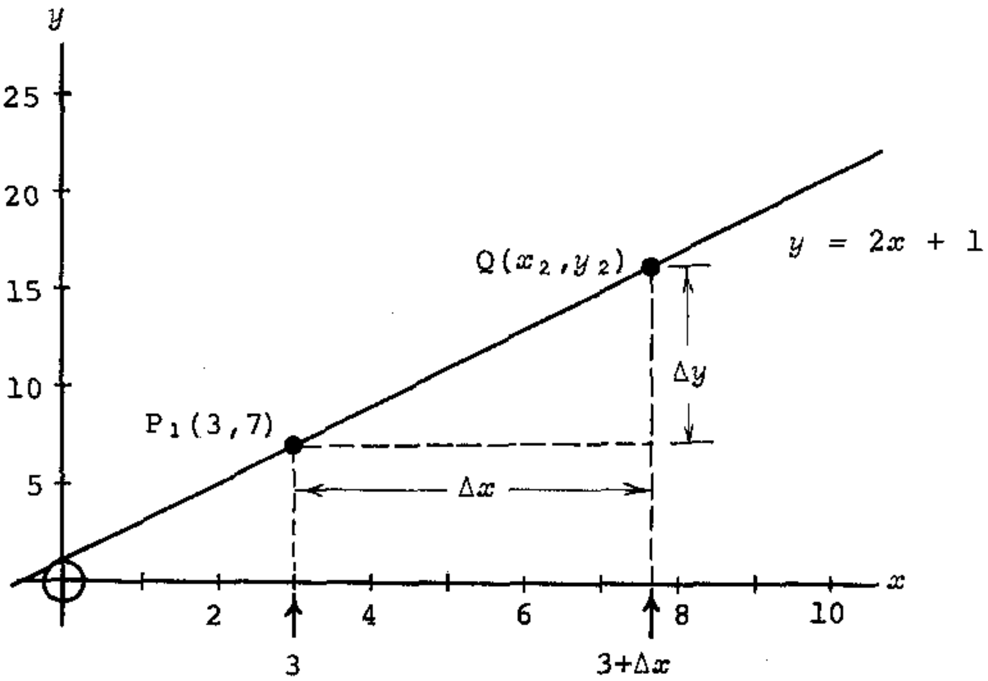

Figure 2

The following table has been constructed with reference to Figure 2, showing the slopes of segments P_1Q for various positions of Q as Q moves towards P_1 along the line:

Δx	Coord's of Q		Slope $P_1Q = \frac{y_2-7}{x_2-3}$
	x_2	y_2	
10	13	27	$\frac{27-7}{13-3} = 2$
5	8	17	$\frac{17-7}{8-3} = 2$
1	4	9	$\frac{9-7}{4-3} = 2$
.1	3.1	7.2	$\frac{7.2-7}{3.1-3} = 2$
.01	3.01	7.02	$\frac{7.02-7}{3.01-3} = 2$
10^{-6}	$3 + 10^{-6}$	$7 + 2 \times 10^{-6}$	$\frac{7+2\times10^{-6}-7}{3+10^{-6}-3} = 2$

The pattern of these results indicates that, no matter how close Q gets to P_1 the slope of P_1Q equals 2, and that the slope of $y = 2x + 1$ AT $P_1(3,7)$ is therefore probably equal to 2.

This can be proved algebraically as follows:

Slope of line at $P_1(3,7) = \lim_{Q \to P_1}$ slope of segment P_1Q,

where Q has coordinates $x_2 = 3 + \Delta x$

$$\begin{aligned} \text{and } y_2 &= f(x_2) \\ &= f(3 + \Delta x) \\ &= 2(3 + \Delta x) + 1 \\ &= 6 + 2\Delta x + 1 \\ &= 7 + 2\Delta x \end{aligned}$$

$\therefore$ slope of line at $P_1(3,7)$

$$\begin{aligned} &= \lim_{\Delta x \to 0} \frac{y_2 - y_1}{x_2 - x_1} \\ &= \lim_{\Delta x \to 0} \frac{(7+2\Delta x)-7}{3+\Delta x-3} \\ &= \lim_{\Delta x \to 0} \frac{2\not{\Delta x}}{\not{\Delta x}} \\ &= \lim_{\Delta x \to 0} 2 \\ &= 2 \quad (\text{"2" is a constant, independent of } \Delta x) \end{aligned}$$

Note that it would be improper to substitute "0" for "Δx" before the second-last line above, since this would lead to the *indeterminate form*, "0÷0".

Exercise:

Do an analysis similar to the above to prove that the 'instantaneous' rate of change of $f(x) = 5x - 2$ at (1,3) equals 5.

Example 2

Prove that the 'instantaneous' rate of change of the linear function

$$f(x) = mx + b$$

with respect to x, at point $P_1(x_1,y_1)$, equals "m".

Solution

The problem is equivalent to proving that the slope of the line $y = mx + b$ at the point $P_1(x_1,y_1)$ equals "m".

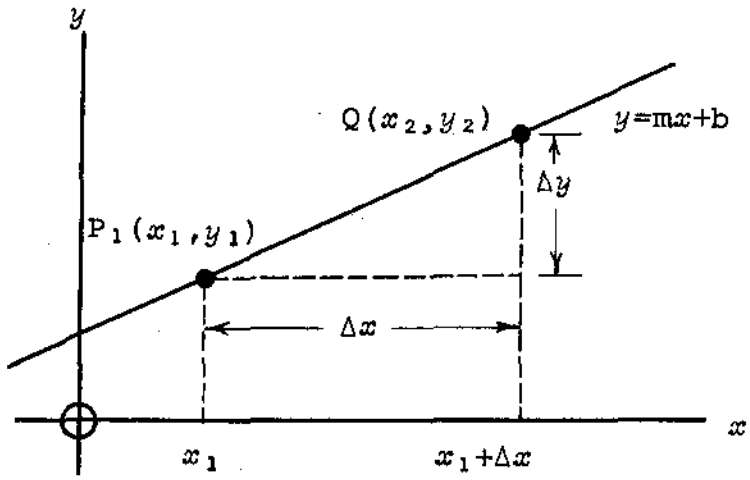

Figure 3

Slope of $y = mx + b$ at $P_1 = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, (see Figure 3)

$$\begin{aligned} \text{where } \Delta x &= x_2 - x_1 \\ &= (x_1 + \Delta x) - x_1 \\ &= \Delta x \end{aligned}$$

$$\begin{aligned} \text{and } \quad \Delta y &= y_2 - y_1 \\ &= f(x_2) - f(x_1) \\ &= (mx_2 + b) - (mx_1 + b) \\ &= m(x_2 - x_1) \\ &= m\Delta x \end{aligned}$$

$$\begin{aligned} \therefore \text{ slope at } P_1 &= \lim_{\Delta x \to 0} \frac{m\cancel{\Delta x}}{\cancel{\Delta x}} \\ &= \lim_{\Delta x \to 0} m \\ &= m \end{aligned}$$

CONCLUSION: THE 'INSTANTANEOUS' RATE OF CHANGE OF A LINEAR FUNCTION EQUALS THE AVERAGE RATE OF CHANGE OF THE SAME FUNCTION, AND BOTH ARE EQUIVALENT TO THE SLOPE OF THE LINE REPRESENTED BY THE FUNCTION.

Notation: "$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$" is abbreviated "$\frac{dy}{dx}$"

read "dee y by dee x", and is called the *derivative of y with respect to x.*

Definition:

The derivative of a function f(x) with respect to x is the 'instantaneous' rate of change of the function with respect to x.

Thus the words "'instantaneous' rate of change" are interchangeable with "derivative" in the foregoing.

II GENERALIZATION TO INCLUDE NONLINEAR FUNCTIONS

Definition:

The *derivative ('instantaneous' rate of change) of a function f(x)* at the point $P_1(x_1,y_1)$ is the limit as Δx tends to zero, of the average rate of change of f(x) with respect to x over the interval $x = x_1$ to $x = x_1 + \Delta x$.

Symbolically,

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$$

The notation "$f'(x_1)$", read "f-primed at x_1", stands for

"the derivative of function f(x), evaluated at $x = x_1$"

OR "the instantaneous rate of change of f(x) with respect to x at $x = x_1$".

Hereafter "rate of change of" will be abbreviated "R/C" and "with respect to" will be abbreviated "wrt".

Graphical Significance of Definition of Derivative

Definitions:

A *secant* to a curve $y = f(x)$ is a straight line cutting the curve at two points.

A *tangent* to a curve $y = f(x)$ is a straight line touching the curve at one point only.

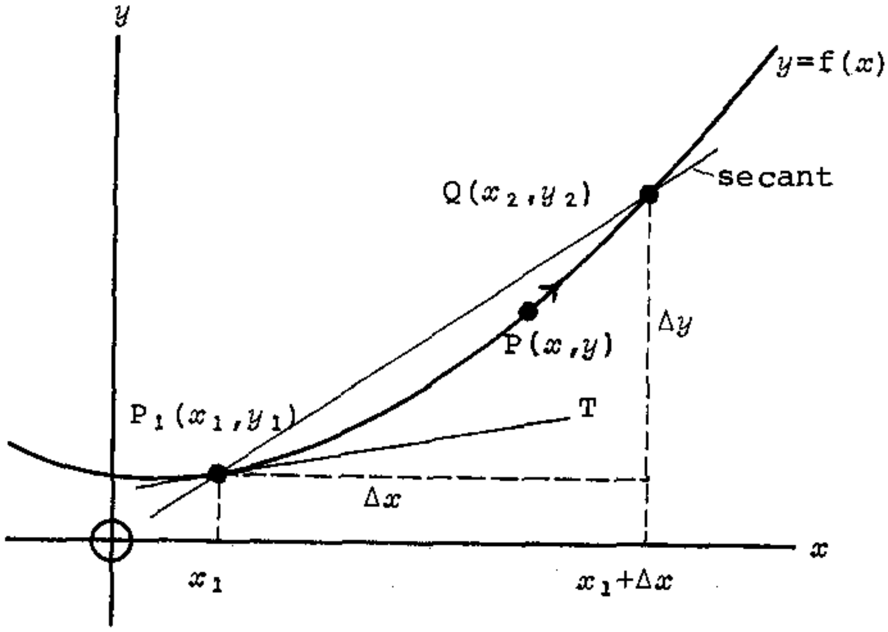

Figure 4

With reference to Figure 4, as point $P(x,y)$ moves up the curve from $P_1(x_1,y_1)$ to $Q(x_2,y_2)$ x changes by Δx, from x_1 to $x_1 + \Delta x$, and y changes by Δy, from $f(x_1)$ to $f(x_1 + \Delta x)$

$$\therefore \text{ average R/C } f(x) \text{ wrt } x = \text{slope of secant } P_1Q$$

$$= \frac{\Delta y}{\Delta x}$$

$$= \frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$$

Now imagine point Q moving down the curve towards P_1. As Q moves towards P_1, the secant P_1Q rotates clockwise and the interval Δx shortens, until, in the limiting position Q coincides with P_1, $\Delta x = 0$, and secant P_1Q coincides with tangent P_1T. Furthermore, the average R/C $f(x)$ wrt x (secant slope) becomes the 'instantaneous' R/C $f(x)$ wrt x (tangent slope).

It should be obvious that the tangent slope at P_1 equals $f'(x_1)$, the derivative at P_1, since the tangent takes the same direction as the curve at P_1. Thus the R/C y wrt x along the tangent line is the same as along the curve at the point of tangency. In fact, when one speaks of the "slope of a curve" one is understood to mean the "slope of the tangent to the curve".

To summarize, the following are equivalent:

(1) 'instantaneous' R/C f(x) wrt x at $x = x_1$

(2) the derivative of f(x) evaluated at $x = x_1$:

$$f'(x_1) = \lim_{\Delta x \to o} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

(3) the instantaneous R/C y wrt x at $x = x_1$, where $y = f(x)$:

$$\frac{dy}{dx} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x} \qquad (\Delta x = x_2 - x_1)$$

(4) $\lim_{Q \to P_1}$ (slope of secant P_1Q)

(5) tangent slope at $P_1(x_1, y_1)$

(6) slope of curve $y = f(x)$ at $x = x_1$

Example 3

Find the 'instantaneous' R/C $f(x) = x^2$ wrt x at $x = 2$.

Solution

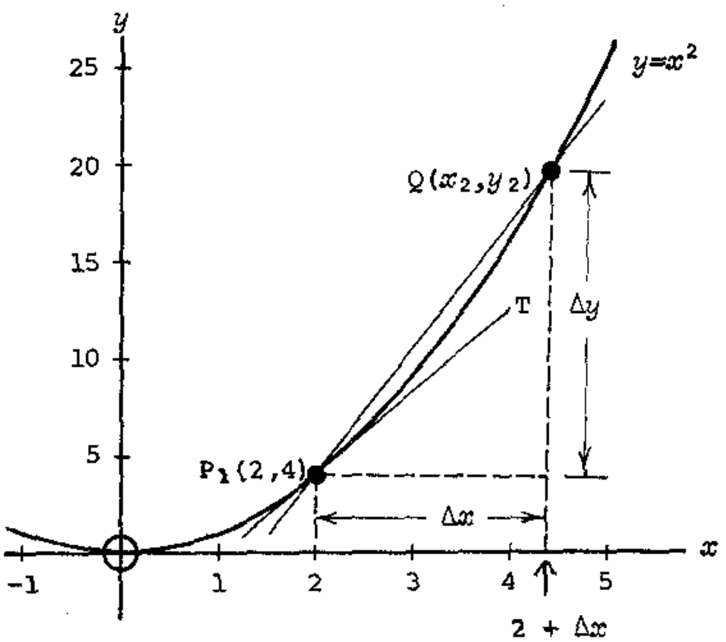

Figure 5

The following table has been constructed with reference to Figure 5, showing the slopes of secant P_1Q for various positions of Q as Q moves towards P_1 along the curve:

Δx	Coord's of Q		Slope $P_1Q = \frac{y_2-4}{x_2-2}$	
	x_2	y_2		
5	7	49	$\frac{49-4}{7-2}$	= 9
1	3	9	$\frac{9-4}{3-2}$	= 5
0.1	2.1	4.41	$\frac{4.41-4}{2.1-2}$	= 4.1
0.01	2.01	4.0401	$\frac{4.0401-4}{2.01-2}$	= 4.01
10^{-6}	$2 + 10^{-6}$	$4+4\text{x}10^{-6}+10^{-12}$	$\frac{4+4\text{x}10^{-6}+10^{-12}}{2+10^{-6}-2}$	$= 4 + 10^{-6}$

The pattern of these results indicates that the slope of secant P_1Q approaches ever more closely to 4 as Q approaches P_1 along the curve, ie, that the tangent slope of P_1 is likely equal to 4.

This will now be proved algebraically:

$$\begin{aligned}\text{Tangent slope at } P_1(2,4) &= f'(2)\\ &= \lim_{\Delta x \to o} \frac{f(2+\Delta x)-f(2)}{\Delta x}\\ &= \lim_{\Delta x \to o} \frac{(2+\Delta x)^2-2^2}{\Delta x}\\ &= \lim_{\Delta x \to o} \frac{4+4\Delta x+(\Delta x)^2-4}{\Delta x}\\ &= \lim_{\Delta x \to o} (4+\Delta x)\\ &= 4\end{aligned}$$

Exercise:

Do an analysis similar to the foregoing to show that the 'instantaneous' R/C $f(x) = 2x^2 + 5$ wrt x at $x = 3$ equals 12.

Example 4 - Power Functions

Definition:

A *power function* is a function of the form $f(x) = x^n$, n a constant.

The derivative of $f(x) = x^n$ at point $P_1(x_1,y_1)$ is

$$f'(x_1) = \lim_{\Delta x \to 0} \frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x_1+\Delta x)^n - x_1{}^n}{\Delta x}$$

It can be shown with the use of the binomial expansion formula, which is beyond the scope of this course, that this limit equals $nx_1{}^{n-1}$, ie,

$$f'(x_1) = nx_1{}^{n-1}$$

Since x_1 can take any value, the subscript on x_1 can be dropped, and the general result for a power function is:

$$\boxed{f(x) = x^n \Rightarrow f'(x) = nx^{n-1}}$$

NOTE that $f'(x)$ is the *derivative function*, ie, $f'(x)=nx^{n-1}$, is a formula for calculating the 'instantaneous' R/C $f(x) = x^n$ wrt x at any point $P(x,y)$.

Example 5

Use the result of Example 4 to obtain the 'instantaneous' R/C $f(x) = x^2$ wrt x at $x = 2$ (cf Example 3).

Solution

$$f(x) = x^2 \Longrightarrow f'(x) = 2x^{2-1}$$
$$= 2x$$

$$\therefore \quad f'(2) = 2(2)$$
$$= 4$$

$\therefore$ 'instantaneous' R/C $f(x) = x^2$, at $x = 2$, equals 4.

Example 6

Find the slope of the tangent to $y = x^3$ at $x = -2.5$.

Solution

$$f(x) = x^3 \Longrightarrow f'(x) = 3x^2$$

$$\therefore \quad f'(-2.5) = 3(-2.5)^2$$
$$= 18.75$$

$\therefore$ slope of tangent to $y = x^3$, at $x = -2.5$, equals 18.75.

NOTE that alternative notations for writing down the result for power functions are:

$$\boxed{y = x^n \Longrightarrow \frac{dy}{dx} = nx^{n-1}}$$

or, simply,

$$\boxed{\frac{d}{dx}x^n = nx^{n-1}}$$

In the latter notation " $\frac{d}{dx}$ ", read "dee by dee x of...", is regarded as an operator, which operates on the function x^n to produce its rate of change, nx^{n-1}.

III STANDARD DIFFERENTIATION FORMULAS

Definition:

To *differentiate* a function is to find its derivative.

The process of differentiating is called *differentiation*.

Trainees are expected to be able to apply the following formulas:

(1) $\frac{d}{dx} x^n = nx^{n-1}$ (power rule)

(2) $\frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$, where "c" is a constant

(3) $\frac{d}{dx} c = 0$, where "c" is a constant

(4) $\frac{d}{dx} (f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

The power rule was developed in the preceding section. Formula (2) may be stated epigrammatically as follows: "The derivative of a constant times a function equals the constant times the derivative".

Proof of Formula 2:

Let $g(x) = cf(x)$

Then, $g'(x) = \lim_{\Delta x \to o} \frac{g(x+\Delta x)-g(x)}{\Delta x}$

$= \lim_{\Delta x \to o} \frac{cf(x+\Delta x)-cf(x)}{\Delta x}$

$= c \lim_{\Delta x \to o} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

$= cf'(x)$

$\therefore \frac{d}{dx} cf(x) = c \frac{d}{dx} f(x)$

Example 7

$$\frac{d}{dx}7x^5 = 7\frac{d}{dx}x^5$$

$$= 7(5x^4)$$

$$= 35x^4$$

Proof of Formula 3:

Let $f(x) = c$.

Then, $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= 0$$

Aside:

Note that if "0" were actually substituted for "Δx" in the second-last line above, the result would be the indeterminate form "0÷0"; however, the process of taking the limit as $\Delta x \to 0$ is not that of simply substituting "0" for "Δx", but rather that of ascertaining the value of an expression as "Δx" tends to "0". (A more advanced or rigorous treatment would include a formal discussion of *limit theory*; this text glosses over many subtleties of the subject.) Note that $0 \div \Delta x = 0$ for any finite value of Δx, no matter how small.

Note that the graph of $y = f(x) = c$ is a straight line, parallel to the x-axis, with slope equal to zero (see Figure 6), consistent with a zero derivative value.

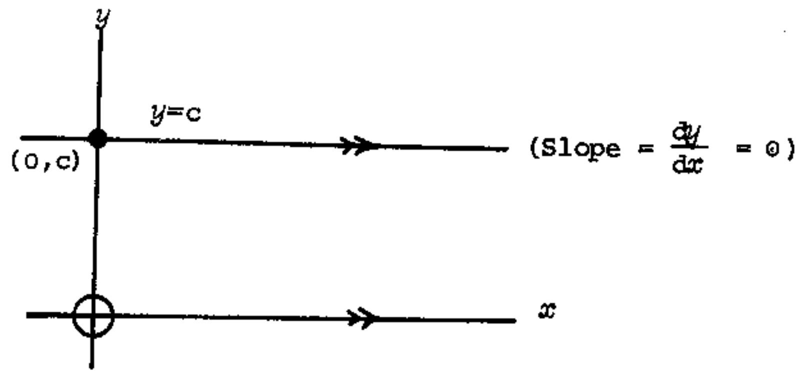

Figure 6

Example 8

(a) $\frac{d}{dx} 8 = 0$

(b) $\frac{d}{dx} (-13) = 0$

(c) $\frac{d}{dx} \pi = 0$

Proof of Formula 4:

Let $h(x) = f(x) + g(x)$

Then, $h'(x) = \lim_{\Delta x \to o} \frac{h(x+\Delta x)-h(x)}{\Delta x}$

$$= \lim_{\Delta x \to o} \frac{[f(x+\Delta x)+g(x+\Delta x)]-[f(x)+g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to o} \frac{[f(x+\Delta x)-f(x)]+[g(x+\Delta x)-g(x)]}{\Delta x}$$

$$= \lim_{\Delta x \to o} \left(\frac{f(x+\Delta x)-f(x)}{\Delta x} + \frac{g(x+\Delta x)-g(x)}{\Delta x}\right)$$

$$= \lim_{\Delta x \to o} \frac{f(x+\Delta x)-f(x)}{\Delta x} + \lim_{\Delta x \to o} \frac{g(x+\Delta x)-g(x)}{\Delta x}$$

$$= f'(x) + g'(x)$$

ie, $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

The proof is similar that

$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$

<u>Example 9</u>

(a) $\frac{d}{dx}[x^3+x^7] = \frac{d}{dx}x^3 + \frac{d}{dx}x^7$ (law (4))

$= 3x^2 + 7x^6$ (law (1))

(b) $\frac{d}{dx}[6x^2-2x^3] = \frac{d}{dx}6x^2 - \frac{d}{dx}2x^3$ (law (4))

$= 6\frac{d}{dx}x^2 - 2\frac{d}{dx}x^3$ (law (2))

$= 6(2x) - 2(3x^2)$ (law (1))

$= 12x - 6x^2$

(c) $\frac{d}{dx}[15x^2+10] = \frac{d}{dx}15x^2 + \frac{d}{dx}10$ (law (4))

$= 15\frac{d}{dx}x^2 + 0$ (laws (2), (3))

$= 15(2x)$ (law (1))

$= 30x$

(d) $\frac{d}{dx}2\sqrt{x} = \frac{d}{dx}2x^{1/2}$ ($\sqrt[n]{x} = x^{1/n}$)

$= 2\frac{d}{dx}x^{1/2}$ (law (2))

$= 2\left(\frac{1}{2}x^{1/2-1}\right)$ (law (1))

$= x^{-1/2}$ or $\frac{1}{x^{1/2}}$ or $\frac{1}{\sqrt{x}}$

Example 10

Find the tangent slope to the curve $y = \sqrt{x}\ (x^2+5)$ at $x = 1$.

Solution

Since the rule for differentiating a product of two functions of x ($\sqrt{x}$ and (x^2+5)) has not been given, the product must first be evaluated:

$$y = \sqrt{x}\ (x^2+5)$$

$$= x^{1/2}\ (x^2+5)$$

$$= x^{5/2} + 5x^{1/2}$$

Then $\frac{dy}{dx} = \frac{d}{dx}(x^{5/2} + 5x^{1/2})$

$$= \frac{d}{dx}x^{5/2} + \frac{d}{dx}5x^{1/2} \qquad \text{(law (4))}$$

$$= \frac{5}{2}x^{3/2} + 5\frac{d}{dx}x^{1/2}$$

$$= \frac{5}{2}x^{3/2} + \frac{5}{2}x^{-1/2}$$

$$= \frac{5}{2}\sqrt{x^3} + \frac{5}{2\sqrt{x}}$$

$\therefore$ at $x = 1$, tangent slope $= \frac{5}{2}\sqrt{1^3} + \frac{5}{2\sqrt{1}}$

$$= \frac{5}{2} + \frac{5}{2}$$

$$= 5$$

ASSIGNMENT

1. Find the tangent slope at $(x, f(x))$ for each of the following functions:

   (i) by evaluating $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

   (ii) by applying the differentiation formulas.

   Include graphs of the functions, and evaluate the tangent slope at $x = 2$ in each case.

   (a) $f(x) = 5x^2 - 2x + 1$

   (b) $f(x) = \frac{2}{x}$

2. Find $\frac{dy}{dx}$:

   (a) $y = 2x^4 - 4x^3 + 15$

   (b) $y = \frac{x^2}{a^2} + \frac{a^2}{x^2}$ where "a" is a constant

   (c) $y = \frac{3}{\sqrt{x}}$

3. Find $f'(x)$:

   (a) $f(x) = x^2 - 6x + 3$

   (b) $f(x) = x^3\ (2x^2-1)$

   (c) $f(x) = ax^2 + bx + c$

   (d) $f(x) = \sqrt[3]{x^2} - 3\sqrt[3]{x} - 5$

4. Find

(a) the 'instantaneous' R/C $y = 2x^3 - 3x^2 - x + 5$ at $x = 2$.

(b) the slope of the tangent to $y = \frac{x+1}{\sqrt{x}}$ at $x = \frac{1}{4}$

(c) the values of x at which the derivatives of x^3 and $x^2 + x$ wrt x are equal. (See Appendix 3 for methods of solving quadratics.)

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